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The Teaching of Approximate Computation*

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Approximate computation is needed in a large proportion of the problems of life since these problems contain approximate data secured from measurements or taken from various tables. The table on the properties of saturated steam in the Handbook of Physics and Chemistry[1] contains 3591 approximate factors, and there are hundreds of similar tables in the various handbooks. In solving problems containing approximate data, the laws of approximate computation should always be followed.

Approximate computation is required in many phases of the mathematics and science taught in the elementary and secondary schools. It is not at all difficult to teach elementary approximate computation in the seventh, eighth, and ninth grades. In fact, it is easier to teach the fundamental facts of this topic in these grades than it is to change the computational habits and concepts of graduate students who have always used "exact" computation. Approximate computation should be taught at the earliest opportunity and then constantly used in all subsequent work in mathematics and science.

Pupils can measure to the *nearest* foot, to the *nearest* 0.1 foot, or to the *nearest* 0.01 ft., according to the graduations on their tape and the care with which it is used, but they cannot measure any length exactly. There are at least 10 sources of error in measuring so simple a thing as the distance between two points several hundred feet apart. Pupils should learn that no one in the whole history of the world has ever made an exactly accurate measurement of any length. The same is true of measurements of time, weight, volume, area, temperature, rotation, latitude, longitude, and the like. The most accurate measurements ever made do not have more than eight significant figures and most of the measurements used in industry have from three to five significant figures.

Figure 1 shows how slight is the relative value of the last figure of a four figure number. If this number had been rounded to 8890 the change in the third bar would have been very small.

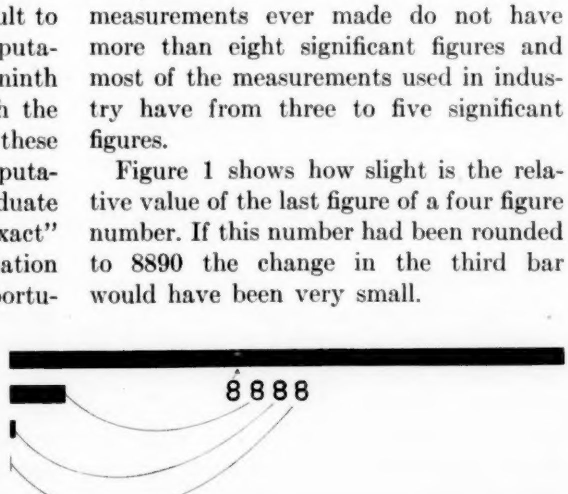

Fig. 1

* The authors have used with permission considerable material from their article "Approximate Computation" in the April 1941 copy of *Education*.

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Figure 2 shows how small is the relative value of *the last three* figures of a six figure number.

Fig. 2

Three and four figure results are *highly satisfactory* for most purposes.

Fundamental Concepts

Elementary pupils should be taught the meaning of "*unit of measurement*" and "*significant figures,*" and *how to round numbers* and *when* and *why* this is necessary. The *unit of measurement* is the smallest unit used in a measurement. The *unit of measurement* in the following measurements is given in the parentheses: 64.32 ft. (0.01 ft.); 43.8 ft. (0.1 ft.); 34.3 lb. (0.1 lb.); 16 lb. 12 oz. (1 oz.); 34°24′16″ (1″); $16\frac{3}{8}$″ ($\frac{1}{8}$″); 6843 mi. (1 mi.); 10.2 sec. (0.1 sec.); 2.1034 in. (0.0001 in.).

If a distance has been measured so accurately that it is known to be nearer to 976.3 ft. than it is to either 976.2 or 976.4 ft., the measurement has four significant figures. *To have all its figures significant every figure of a number except the last must be correct and the error in the last figure must not be greater than one-half the unit of measurement.*

If the measurement 26.8 in. has three significant figures it must be between 26.75 in. and 26.85 in. If the measurement 165.37 ft. has five significant figures it must be between 165.365 ft. and 165.375 ft.

Zeros that are the result of correct measurement are significant.

If the measurement 300 ft. is correct to the nearest foot or lies between 299.5 ft. and 200.5 ft. the measurement has three significant figures and the two zeros are both significant. Zeros should not be written after a decimal point in a mixed decimal unless they are significant. You can be sure that the measurement 26.00 ft. has four significant figures and that the measurement 3.0000 in. has five significant figures.

Zeros used to give the correct place value in a rounded number are not significant. If 186284 is rounded to 186000 the three zeros are not significant. Zeros are also used before significant figures in a decimal fraction to give correct place value. Zeros used in this way are never significant. If the three figure measurement 86.3 mm. is changed to 0.0000863 km., the zeros are not significant.

Various devices have been suggested to indicate when zeros are significant, but these are not necessary. In a set of measurements one can usually tell which zeros are significant.

In the set of measurements at the right it is safe to assume that the zeros are significant. Like people, zeros may be judged by the company they keep. If one sees such measurements as 6,000,000,000,000 miles or 186,000 miles one may usually assume that these are rounded numbers, and that the zeros are not significant.

3.000	in.
4.123	"
1.300	"
2.177	"
10.600	"

Pupils who secure their own data through measuring will always know which zeros are significant and no author should ever give a rounded number in a problem without telling the number of significant figures. Example: The speed of light, correct to three figures, is 186,000 miles per second. How long will it take light to go 7,927 miles? Or the speed of light is 186,300 miles per second (4 s. f.) etc.

To find how many significant figures there are in a measurement when decimal measures are not used, express the measurement in terms of the *smallest unit of measurement used.* Thus, 46° has two significant figures, 46°0′ or 2760′ has four significant figures, and 46°0′0″ or 165600″ has six significant figures; 9 in. has only one significant figure, but 9 0/64 in. or 576/64 has three; 68 lb. 4 oz. or 1092 oz.

has four significant figures, and 9 ft. has one significant figure but 9 ft. 0 in. or 108 in. has three significant figures and 9 ft. 0/32 in. or 3456/32 in. has four significant figures.

When changing measurements containing common fractions to decimals, retain only the same number of significant figures as there are in the original data. Thus $6\frac{3}{8}$ in. or 51/8 in. (two sig. fig.) is not equivalent to 6.375 in. (four sig. fig.) and $2\frac{2}{3}$ in. or 8/3 in. (one sig. fig.) is most certainly not equal to 2.667 in. (four sig. fig.). Common fractions are very unsatisfactory in real life computation because of their lack of accuracy, and the fact that they cannot be used conveniently with slide rules, computing machines, logarithms or in other tables. They are also less convenient than decimal fractions to type and print and are more difficult to use in pencil computation. Because of these and a number of other reasons decimal fractions are rapidly replacing the old fashioned common fractions.

Rounding Numbers

All elementary texts teach pupils *how* to round numbers; but they do not teach them *when* to round numbers or *why* the answers to certain problems must be rounded, or *how to determine* the number of figures to retain when rounding these answers.

The following rules may be used for rounding numbers:

1. If a measurement or answer, correct to a certain number of significant figures is to be rounded to a smaller number of significant figures, the figures that are dropped should be replaced by zeros. When the figures that are dropped are located to the right of the decimal point, the use of zeros is not necessary.
2. If the first of the figures that are to be dropped is 5, 6, 7, 8, or 9, the last figure retained should be increased by 1. (Note: The term *digits* may be used in place of *figures* in discussing approximate computation.)

Adding and Subtracting Approximate Numbers

In a set of measurements like those shown in *A*, each measurement should be given to the same unit; in this case the tenth of a foot is the unit of measurement since each measurement is made to the nearest tenth of a foot.

A	*B*	*C*
42.6 ft.	26.31 ft.	9 8/16
80.0 "	30.00 "	4 3/16
76.3 "	16.36 "	5 1/16
8.0 "	41.30 "	3 0/16
206.9 ft.	113.97 ft.	21 12/16

In the set *A*, the zeros are significant and must be written as shown. In set *B*, the unit is 0.01 ft. The zeros are significant and cannot be omitted. In set *C*, the unit of measurement, 1/16 in., is clearly indicated. The 8/16 in. should not be reduced to $\frac{1}{2}$ in. The 0/16 is unusual but is necessary to show the same unit has been used for each measurement.

The last figure of each measurement in *A* and *B* is a nearest figure and may be slightly too large or slightly too small. The measurements in *C* may each be as much as 1/32, or one half of the unit of measurement, too large or too small. Because of this the last figure of the answers to *A*, *B*, and *C* may not be significant but in practical work the best rule is to retain the full answer. In subtracting approximate numbers it is better to retain the full answer as shown in *D*. Quantities to be added or subtracted should be measured to the same unit, or if not, should be rounded to the same unit before they are added or subtracted.

D	*E*	*F*
86.24 in.	251. ft.	64. ft.
23.41 "	32.641 "	12.376 "
	4.37 "	
62.83 in.	120.5 "	

Unfortunately, examples in addition and subtraction like *E* and *F* may still be

found in text books and obsolete examinations. Examples of this type do not occur in real life. Professor Reeve says: "Not only is an example like $12.7-4.0396$ a nonessential but it is an evidence of educational ignorance."[2]

Writing of similar examples found on so-called standard tests, Professor Upton says: "How is a teacher to keep her balance when presumably authoritative tests give problems like the above?"[3]

Examples E and F *violate the fundamental rule of approximate computation that all measurements to be added or subtracted should be measured to the same unit.* Professor A. R. Cullimore says: "The frequent habit of carrying results to a greater number of significant figures than the data warrants come perilously near to lying with figures."[4] The teacher who annexes zeros in examples like E and F and thus makes people believe that rough data is very accurate data is no longer "perilously near"; he has arrived. It is *never* permissib'e to annex zeros in examples like E and F. The measurement 251 ft. in E may be anything between 250.500 ft. to 251.500 ft. Therefore, 251.000 ft. is only one of a possible thousand values, each of which would produce a different answer.

A measurement like 3 in. is a very rough measurement, while 3.0000 in. is a very precise (small unit) and accurate (five sig. fig.) measurement. In no sense are 3 in. and 3.0000 in. equivalent. If the 3.0000 in. has been measured correctly to the nearest .0001 in. the zeros are significant and no practical measurer would omit them.

Criteria for Accuracy

The measurements 4832 ft., 48.32 ft., and 0.4832 ft., are all equally accurate since they have the same number of significant figures but the last measurement is the most precise since it has a much smaller or more precise unit of measurement. An instrument of precision was needed to make the last measurement. A measurement or the answer to a problem must have the decimal point correctly located but the location of the decimal point does not indicate the accuracy of a measurement or the accuracy of an answer. Many texts instruct pupils to carry out answers to a stated number of decimal places. This is not a satisfactory criterion and often leads to serious errors. The only safe criterion is *carry the answer out to the number of significant figures justified by the data.*

The criterion for accuracy that is easiest to apply in the ordinary computation used in elementary schools and in high schools, is the number of significant figures. As 36 mm. and 0.000036 km. each have two significant figures, they are of the same degree of accuracy. *A measurement having three significant figures, or three-figure accuracy is more accurate than a measurement having two significant figures or two-figure accuracy.* An approximate number having four significant figures, or four-figure accuracy is more accurate than one having three significant figures, or three-figure accuracy. An approximate number having $(n+1)$ significant figures is more accurate than one having only (n) significant figures.

If two measurements have the same number of significant figures the one that begins with the larger digit is the more accurate. Thus, 8.76 ft. is more accurate than 342. ft., 4.37 ft., 0.135 ft., or 63.8 ft. The measurement 99.9 ft. is very near to four-figure accuracy $(99.9+0.1=100.0)$, while the measurement 10.0 ft. has just gotten into the three-figure class $(9.0+0.1)$. If there is an error of 0.05 ft. in each of the last two measurements the error in the first is 0.05 in 99.5 or 1 in 1990 while the error in the second is 0.05 in 10.0 or 1 in only 200. It is easily seen that the error in the second measurement is far more serious than the error in the first. The *ratio of the error in the measurement to the measurement itself is called the relative error.* Where a rigorous criterion for accuracy is needed the relative error should be used.

Rules for Multiplying and Dividing Approximate Numbers

When two approximate numbers are to be multiplied or divided, the following rules should be used:

1. *If two approximate numbers have the same number of significant figures, multiply the numbers and round off the product to the same number of figures as there are in each factor.* The last figure of the answer will not always be significant, but this rule is satisfactory for all elementary work and is usually followed in scientific work.
2. *If one of two approximate numbers has more significant figures than the other, first round off the more accurate approximate number so that it contains one more significant figure than the less accurate approximate number. Then multiply the numbers and round off the product to the same number of figures as there are in the less accurate factor.* A product can never have more significant figures than there are in the least accurate of the factors used in the computation. In some cases it may have one less significant figure, but for elementary work the rules given above would be used.
3. *If the two approximate numbers have the same number of significant figures, carry the quotient out to one more figure than is contained in each of the given numbers. Then round off the quotient so that it contains the same number of figures as there are in each of the given numbers.*
4. *If the dividend and divisor are such that one of them has more significant figures than the other, first round off the more accurate number so that it contains one more significant figure than the less accurate number. Then divide and carry the quotient out to one more figure than is contained in the less accurate of the two numbers. Finally round off the quotient to the same number of figures as are found in the less accurate number.*

When the above rules for dividing approximate numbers are used, the last figure of the quotient will not always be significant but these rules should be used for all elementary work.

A rough demonstration like the one at the right will help to show the reasonableness of the laws for multiplication. The uncertain figures are in boldface. All figures obtained by using these uncertain figures are also shown in boldface.

```
   8.65
   7.43
 ------
   2595
  3460
 6055
 -------
 64.2695
Best ans. 64.3
```

It is evident that it would be foolish to retain more than one uncertain figure. This device may be used for multiplication or division. Pupils may use red pencil for the uncertain figures. A second demonstration is to take the product of the two lower limits (8.645×7.435) and the two upper limits (8.655×7.435) and round the answers to three figures. If all the possible combinations between these limits are taken there will be 11×11 or 121 different answers. Any one of these may be *the* answer. However, if rules 1 and 2 are followed the best answer will be obtained in a very large per cent of the cases.

5. *In the actual work of dividing two approximate numbers, it is sometimes necessary to annex zeros to the dividend in order to secure in the quotient the number of figures warranted by the original data.* In the example $32.4 \div 81.96$ the answer should be carried out to four figures and rounded back to three (answer 0.395). To secure this answer it was necessary to annex several zeros. If the dividend is more accurate than the divisor the original figures may be retained. It is *never* permissible to annex zeros in addition or subtraction.
6. *The answer in square root should contain the same number of significant figures as there are in the approximate number whose root is sought. The square root of 64 is 8.0 and the square root of 64.00 is 8.000. It is better to carry the answer out one figure farther than warranted and then round back as was done in division.*
7. *When used in multiplication, division,*

and square root, the rules of approximate computation should be applied only to the approximate factors. The 4, 3, 2, 6 and 4 in the following formulas are exact numbers:

$$V=4/3\pi r^3, \qquad A=\tfrac{1}{2}ab,$$
$$V=h/6(B+4m+t)$$

Pupils must be trained to differentiate between exact and approximate numbers.

Exact Numbers

1. *Numbers obtained by counting are considered exact.* This is especially true when the elements counted are practically identical such as six nickels, eight 1 in. steel balls, twelve standard eggs of the same grade. When the elements counted are not identical the "measurement" may be *wildly approximate for some purposes.* If a new development contains 86 houses all built from the same set of plans and costing $7600 each, the 86 may be considered an exact number. However, if there are in a town 86 houses ranging in value from a mansion costing $95,000 down to a shack costing $800, the 86 is exact only in a "census" sense.

   Large numbers obtained by counting should be carefully checked to see that counting produced no error. If we read that a certain city has 1,276,385 inhabitants we may be sure, for a number of good reasons, that little confidence can be placed in the last two or three figures.
2. *Small whole numbers in various formulas are almost always exact.*
3. *Hypothetical measurements may be considered exact.* If the sides of a square were exactly 2 in. the perimeter would be exactly 8 in., the area would be exactly 4 sq. in. and the diagonal $2\sqrt{2}$ in. The $\sqrt{2}$ in this case could be carried out to any desired number of significant figures. It might be well to note, however, that such a figure has never existed and no one could construct it or measure its sides exactly, if it did exist.

Approximate Numbers

1. *All measurements of all kinds are approximate* (counting not included).
2. Ratios of measured results are approximate.
3. Many numbers, or ratios, such as $\frac{2}{3}$, π, e, $\sqrt{3}$, tan x, etc., cannot be expressed exactly by an ordinary mixed decimal or decimal fraction. *When the first N figures of such a number are taken as a satisfactory approximation, the number thus obtained is approximate.*
4. *All rounded numbers are approximate.* The answer to any problem in which approximate data are used is approximate and must be correctly rounded.
5. *Practically all the numbers taken from various handbook tables are approximate.* There are hundreds of such tables and some of these tables contain thousands of approximate numbers.
6. It is fairly safe to assume that practically all mixed decimals and decimal fractions are approximate numbers.

Miscellaneous Rules and Suggestions

When such approximate numbers as π, 0.7854, 1.732, ($\sqrt{3}$) or any of the thousands of physical constants found in the various tables are used in a formula or problem, round off the value found in the table to one more significant figure than is contained in the least accurate of the other approximate numbers in the formula. This agrees with the rules for multiplying and dividing approximate numbers.

When finding the products of three or more approximate numbers or when squaring or cubing an approximate number follow the rules for *multiplication.* If possible use the more accurate factors first. Somewhat better results will be obtained if one more figure is retained in the partial products than will be retained in the final answer. A similar rule may be used in division or an example in which both multiplication and division are needed.

In this vital matter of approximate com-

putation in the elementary schools and the high schools *we need simple consistent rules that can be easily applied.* Statistical and other refinements are matters for the college and graduate school. The rules given in this article form a safe foundation on which the graduate school may build. It is true that by using the rules given in this article, now and then a final figure that is not significant will be retained. This is not serious and will not conflict with work in science where it is common practice to retain a final figure that is on the "ragged edge."

Many of our modern text books are highly inconsistent in their treatment of approximate computation. Duane Roller says, "Most text books discuss the concept of significant figures but fail to take it into account in stating problems of a quantitative nature thus making it almost impossible for the student to use the idea of significant figures in solving problems."[5] A few modern texts give a fairly good treatment of approximate computation, but in succeeding chapters and in their answer books, they fail to follow the rules for approximate computation that they developed.

The data in any given problem in mathematics or science should be consistent, and should clearly indicate the accuracy desired in the answer. No answer book should give and no teacher should allow more figures in a final answer in multiplication, division or square root than there are in the least accurate item.

Approximate computation is not rough, careless or slipshod work. It is careful, intelligent computation that produces honest answers. The answers obtained are the best answers that can be secured from the given data.

Approximate Computation in the Elementary School

Approximate computation should be taught in grades 7, 8, and 9 because of the following reasons.

1. It is the only "real life" computation for practical or applied problems in which measurements of any type are part of the data.
2. It gives pupils a definite criterion for rounding answers in multiplication. That is, it gives the why, when, and how for rounding.
3. It gives pupils a criterion for telling at a glance how far to carry out any problem in division.
4. It gives pupils a criterion for determining the number of figures to retain when using such numbers as 3.14159, 0.785398 or 0.3183098, metric conversion factors—1 km. = 0.6214 mi., approximate values such as 1.25 cu. ft. = 1.00 bu., specific gravities—lead equals 11.34, tangents and the like.
5. It gives pupils a criterion for telling how far to carry out the answer to any example in square root.
6. It gives the pupil a criterion for correctly using in a problem data of differing degrees of accuracy.
7. It gives pupils a criterion for telling how far to carry out the answer to an equation, and how to check this answer.
8. It gives the pupils a criterion for telling how far to carry out the answer to any written problem in applied mathematics or science.
9. It prepares pupils for such approximate methods of computation as the use of logarithms, the slide rule, graphic solutions, square root tables, compound interest tables and the like.
10. It enables pupils to use our old measuring units along with decimal factors and retain the proper number of significant figures.
11. It eliminates ragged decimals in addition and subtraction.
12. It enables pupils to tell when they may annex zeros—only in division and roots—and how many zeros they may annex.
13. It gives pupils a criterion for working with fractional measurements, and compound numbers. They will never

make the mistake of assuming that 4½ in. always equals 4 32/64 in., and they will know that if they reduce 4 32/64 in. to 4½ in., they reduce the indicated accuracy of the measurement from three significant figures to one significant figure. They will know that 4 in. has only one significant figure, while 4 0/64 in. has three.

14. It saves a considerable amount of time spent in useless calculation and gives better results (this is especially true when abridged multiplication or division is used).

Exercises in Approximate Computation

1. Round to three significant figures. Answers in parentheses: 87.46, (87.5); 92.54, (92.5); 3.1416, (3.14); 2956.5, (2960).
2. Round to four significant figures: 3.14159, (3.142); 0.78539, (0.7854); 219352, (219400).
3. Round to one significant figure: 3956.5, (4000.); 59.86, (60).
4. 8.64×7.26 (multiply and round to three significant figures), (62.7); 14.36×18.29, (262.6); 42.9×86.5 (3710, zero not sig.); 28.56×8.23, (235.).
5. 8.7 ft.$\times$3.1416, (8.7$\times$3.14$=$27 ft.); 424.86×8.43, ($424.9\times 8.43=3580$) (zero not sig.).
6. $57.6\div 3.62$, (15.9); $43.86\div 18.23$, (2.406); $29.8\div 3.14159$, ($29.8\div 3.142=9.48$); $63.1\div 89.73$, (0.703).
7. Find the area of a trapezoid with $b_1=26.34$ ft., $b_2=18.96$ ft., and $a=8.34$ ft. Ans.$=$189 sq. ft.
8. Find the area of a triangle with $b=24.78$ ft. and $a=19.26$ ft. (Ans. 238.6 sq. ft.)
9. Give the value of π that should be used, and the circumferences of the circles having the following diameters: 2.4 ft. (3.14, 7.5 ft.); 22.81 ft. (3.1416, 71.66 ft.); 23.6 in. (3.142, 74.2 in.); 183.59 ft. (3.14159, 576.76 ft.).
10. Find the square root of 81.00 (9.000); 6.2500 (2.5000); 3.000 (1.732); 64 (8.0).
11. What is the volume of a conical pile 32.76 ft. in diameter and 4.941 ft. high?
    Solution: $V=\frac{1}{3}\times 3.1416\times(16.38)^2\times 4.941=1388$ cu. ft.
12. The side of an equilateral triangle is 12.00 ft. What is its area? $A=\dfrac{S^2}{4}\sqrt{3}$.
    Solution: $(12.00)^2=144.0$
    $144.0\times\frac{1}{4}\times 1.7321=62.36$.
    Note: Since 12.00 has four significant figures it is necessary to retain five figures in $\sqrt{3}$. The last figure in the answer may not be significant.

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The Foundation Stones of the Number System

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A famous mathematician has said that the natural numbers 1, 2, 3, $\cdots$, are a gift to us from the deity. Since the whole of our number system is based upon them, it is fitting that they should be studied in all possible ways and that such investigations should be regarded as fundamental in mathematics. One way to begin such a study is to break the integers down by factoring them, to see what they are made of. When we attempt to do this we find certain integers that cannot be factored (except in a trivial way); these are called *prime numbers* and the first few of them are clearly 2, 3, 5, 7, $\cdots$. It is immediately obvious that any other natural number can be expressed as the product of such primes (some of which may be repeated); for example, $60=2^2\cdot3\cdot5$. In addition it is true, though not obvious, that this expression is unique. It therefore seems appropriate to think of these prime numbers as the foundation stones of our number system and we shall devote this paper to some of their properties.

A simple process for constructing a table of primes was invented long ago by Eratosthenes (ca. 200 B.C.). To see how this is done we shall make a table of the primes that are less than 40. First we write down all the numbers from 1 to 40:

1	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~

Then we begin by striking out all the multiples of 2 except 2 itself, that is, we strike out every other number beginning with 4. Next we strike out the multiples of 3 beginning with 6 and we repeat this operation for the primes 5, 7, etc. Certain numbers, such as 6, 10, $\cdots$ will be struck out more than once but there is no objection to this. If we do this for all the primes that are less than $\sqrt{40}$ we will get rid of all the composite numbers in our table. For it is obvious that a composite number must be divisible by a prime factor that is less than the square root of that number. In our table we can stop after we have struck out the multiples of 5 and the numbers that are left are the primes that we desired to list. Of course 2, 3, 5 must be included in this list; on the other hand, the number 1, which is also listed, must be left out since it is not usually counted as a prime. This simple process is known as "the sieve" and it is used, with variations, even today for the construction of such tables.

There are occasions when one wishes to know how many primes there are that are less than a certain number. If a table of primes is available we need only count them; for example, we see that there are 12 primes less than 40. We shall now show how it is possible to count the primes less than 40 without actually writing them down. To do this we count the numbers in our table that were crossed out, that is, those numbers $\leqq 40$ that are multiples of 2, 3, 5. Now there are obviously 20, 13, 8 multiples of these primes, respectively, that are $\leqq 40$ and one might think that the required number is $20+13+8$. Of course this is wrong since we have here counted twice the multiples of $2\cdot3=6$, $2\cdot5=10$, $3\cdot5=15$. Since there are 6, 4, 2 multiples $\leqq 40$ of these numbers respectively, our next attempt at an answer is $20+13+8-6-4-2$. But there is now *one* multiple of $2\cdot3\cdot5=30$ that has not been counted and the correct result is thus

$$20+13+8-6-4-2+1=30.$$

Hence there are $40-30=10$ numbers $\leqq 40$ *not* divisible by 2, 3, 5. To this we must add 3 for the primes 2, 3, 5 and subtract 1

for the number 1 which is not counted as a prime. Thus the required number of primes is 12.

The method we have just explained can be applied to find the number of primes below any given limit N. It is only necessary to count the numbers $\leqq N$ that are divisible by one or more of the primes $\leqq \sqrt{N}$ and this can be done as in our example. If the number of primes for which we have to sift out in this way is large the procedure evidently becomes very laborious and labor-saving devices have been invented to improve the situation. We shall illustrate one such device by finding the number of primes less than 300. The primes $\leqq \sqrt{300}$ are 2, 3, 5, 7, 11, 13, 17 but we first sift out for those $\leqq \sqrt[3]{300}$, namely, 2, 3, 5. It is easy to see that the number of numbers left after this has been done is

$$300-(150+100+60-50-30-20+10)=80.$$

Now we must sift out the multiples of 7, 11, 13, 17 that have not already been ousted, that is, that are *not* multiples of 2, 3, 5. Such a number cannot be divisible by more than *two* primes (since the product of three primes, each $>\sqrt[3]{300}$, is >300) and it is easily seen that the only such numbers are the following:

	Total
7, 11, 13, 17; 7^2, 11^2, 13^2, 17^2	8
$7\cdot 11$, $7\cdot 13$, $\cdots$, $13\cdot 17$	6
$7\cdot p$, p a prime >17, where $7p<300$, i.e. $p<42$	6
$11\cdot p$, p a prime >17, where $11p<300$, i.e. $p<27$	2
$13\cdot p$, p a prime >17, where $13p<300$, i.e. $p\leqq 23$	2
$17\cdot p$, p a prime >17, where $17p<300$, i.e. $p\leqq 17$	0
Total	24

The last four entries were found from our small table of primes. After all this sifting has been done, there are $80-24=56$ numbers left and the desired number of primes is therefore $56+7-1=62$.

The properties of prime numbers that we shall now discuss have to do with the set of primes as a whole, rather than with individual samples. The big problem here is to see how the primes are distributed amongst the rest of the natural numbers. To help in discussing this question it is customary to introduce the notation $\pi(N)$ for the number of primes that are $\leqq N$; thus $\pi(300)=62$. The study of $\pi(N)$ as a function of N is the central subject of investigation in this branch of mathematics and we have just seen how one may gather experimental data about it. The following table gives some values of $\pi(N)$; the more spectacular entries were computed by methods entirely similar to the one we have explained.

N	(N)
100	25
1000	168
10^4	1,229
10^5	9,592
10^6	78,498
10^7	664,579
10^8	5,761,455
10^9	50,847,478

The first theorem to be mentioned here was proved by Euclid (ca. 300 B.C.) who showed that the set of primes is infinite, so that $\pi(N)$ becomes infinite as N does so. The proof is extremely simple. For, suppose that $p_1, p_2, \cdots, p_n$, were all the primes there are. Then we consider the number $p_1, p_2 \cdots p_n+1$ which is surely not divisible by any of these primes; since it is >1, however, it must have a prime factor and this prime is different from the primes $p_1, p_2, \cdots, p_n$. Thus we reach a contradiction.

We now know that $\pi(N)$ increases without limit as N becomes infinite. The next question is to see how $\pi(N)$ behaves for large values of N. One way of getting an idea on this point is to consider the proportion of numbers $\leqq N$ that are primes and we have therefore tabulated the values of

the ratio $\pi(N)/N$ for the same values of N as in our first table.

N	$\pi(N)/N$	$\frac{\pi(N)}{N} \cdot \log_{10} N$
10^2	0.25	0.50
10^3	0.168	0.504
10^4	0.1229	0.4916
10^5	0.09592	0.4796
10^6	0.07850	0.4710
10^7	0.06646	0.4652
10^8	0.05761	0.4609
10^9	0.05085	0.4577

The first thing suggested by this table is that $\pi(N)/N$ approaches zero as N becomes infinite; it is not hard to prove this by the methods we have explained for counting primes. Upon closer inspection the tabulated values of $\pi(N)/N$ are seen to be nearly inversely proportional to 2, 3, 4, $\cdots$, 9, that is, to the common logarithms of the corresponding values of N. To check up on this we have tabulated the values of the product $\frac{\pi(N)}{N} \log_{10} N$ in another column. From these values one might conjecture that the product just mentioned remains nearly constant as N increases, or indeed that it approaches a limit as N becomes infinite. This guess was first made by Legendre in 1798, but it was not proved to be so until 1896. In fact it turns out that

$$\lim_{N\to\infty} \frac{\pi(N)}{N} \cdot \log_{10} N = \log_{10} e = 0.4343,$$

or, expressed in terms of natural logarithms,

$$\lim_{N\to\infty} \frac{\pi(N)}{N/\log N} = 1.$$

This result, usually known as "The Prime Number Theorem," is the fundamental theorem here. It was proved independently by Hadamard and de la Vallée Poussin, who based their proof on the work of Riemann (1860) and used the methods of the theory of functions of a complex variable. Even today this theorem cannot be proved by "elementary" means.

The meaning of the prime number theorem is that the elementary function $N/\log N$ is an approximation to our function $\pi(N)$ for large values of N, in the sense that the ratio of the two functions approaches unity as a limit. It is of course a remarkable fact that so erratic a function as $\pi(N)$ can be approximated by such a simple function in this manner. The reader might amuse himself by using the theorem to determine approximately how many primes there are less than 10^{10}, or less than a trillion. As an example of how the theorem can be applied to theoretical questions we shall compare the number of primes less than N with the number of primes less than $2 \cdot N$. We see at once that

$$\lim_{N\to\infty} \frac{\pi(2N)}{\pi(N)} = \lim_{N\to\infty} \frac{2N/\log 2N}{N/\log N}$$
$$= \lim_{N\to\infty} 2 \cdot \frac{\log N}{\log N + \log 2} = 2.$$

This means that, if N is large enough, then $\pi(2N)$ is very nearly twice $\pi(N)$; in other words there are "approximately" as many primes between N and $2N$ as there are primes less than N. The reader can prove in a similar way that the number of primes less than N^2 is "approximately" equal to $\frac{N}{2} \cdot \pi(N)$.

Our discussion has necessarily been brief, but it is hoped that the reader will have gotten some notion of the properties of the set of prime numbers considered as a whole. One object of the paper was to point out how relatively easy it is to get numerical data about this important set of numbers and how laboratory work of this kind has led mathematicians to make guesses that were later proved by much more complicated methods. There are

many books that the reader can consult for more information; the following two are readily available:

G. H. Hardy and E. M. Wright, "An Introduction to the Theory of Numbers" (Oxford, 1938), chapters 1, 2, 22.

A. E. Ingham, "The Distribution of Prime Numbers" (Cambridge Tract No. 30, 1932).

In addition one should mention:

D. N. Lehmer, "Factor Table for the First Ten Million," "List of Prime Numbers from 1 to 10,006,721" (Carnegie Institute Publications, 1909, 1914); the introductions to these tables contain much interesting information.

Engineering Students versus Other Students in Freshman College Mathematics

By M. C. Bergen
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The writer had the pleasure, a few years ago, of conversing with the dean of liberal arts of a middle-western university. The dean made the statement that he, the writer, would discover in the course of time that future engineers make the poorest students of college mathematics. The statement seemed rather unfair to engineers considering the fact that so much has been said about the need for mathematics in the field of engineering, and that only students who have ability in mathematics ought to choose engineering for a profession. It was then and there decided that the writer would in the future keep a file of all students in his classes, such students to be classified as to the curriculums in which they were enrolled. A comparison could then be made of their achievement. Of course, a few more years of data collecting would result in a greater distribution for each curriculum, but the strong desire to see the comparisons has prompted the writer to publish the data now.

The Morgan Park Junior College offers only the traditional courses in mathematics beginning with college algebra and continuing through the integral calculus. Since only a very few non-engineering students enter courses above trigonometry, the data for analytic geometry and the differential and integral calculus cannot be of much use in this study. For that reason only data for students in college algebra and trigonometry are included here. These two courses take in all engineering students, and all pharmacy students. The commerce curriculum requires one year of a laboratory science or mathematics. The pre-medical and pre-dental courses require trigonometry, while the liberal arts students must take one semester either of psychology or mathematics. There is no mathematical requirement of pre-legal students. The limited departmental offerings at the school, however, make it almost necessary that many students elect mathematics in order to qualify for graduation.

All mathematics courses at the Morgan Park Junior College are of the same content. Engineering, pre-legal, medical students, etc., take the same course. Such a situation lends itself well to this study since the same achievement is expected of all.

Pre-medical and pre-dental students are listed together under one group for this study, namely, pre-medical, because the two curriculums are essentially the same.

College Algebra

Table I shows the distribution of grades in college algebra in the various curriculums, together with the per cent for each. The number of students registered in the pre-medical, pre-legal, and pharmacy courses is rather small compared with the others so that perhaps the large per cent of 'A' students in the pre-legal course is distorted especially since no mathematics is required of them. However, in the curriculums in which there are large enrollments the liberal arts curriculum shows the greatest per cent of 'A' grades (12.0 per cent). The engineering curriculum is second (9.7 per cent) among the three large groups and third for all groups.

At the other end the lowest number of failures is to be found in the liberal arts curriculum (8.0 per cent). Next in order come the pre-legal (12.5 per cent), commerce (16.5 per cent), engineering (17.2 per cent), pre-medical (20.8 per cent), and pharmacy (56.3 per cent) curriculums.

Table II shows the comparison among

TABLE I

Comparison of Grades Made in College Algebra

Final grade	Curriculum												Total	
	Engineering		Liberal Arts		Commerce		Pre-medical		Pre-legal		Pharmacy			
	No.	Per cent	No.	Per cent	No.	Per cent	No.	Per cent	No.	Per cent	No.	Per cent	No.	Per cent
A	14	9.7	12	12.0	5	5.5	1	4.2	2	12.5	—	—	34	8.7
B	18	12.4	26	26.0	13	14.3	5	20.8	1	6.3	1	6.3	64	16.3
C	53	36.6	31	31.0	30	33.0	4	16.7	7	43.8	5	31.3	130	33.2
D	25	17.2	17	17.0	12	13.3	8	33.3	3	18.8	—	—	65	16.6
E	25	17.2	8	8.0	15	16.5	5	20.8	2	12.5	9	56.3	64	16.3
Drop	10	6.9	6	6.0	16	17.6	1	4.2	1	6.3	1	6.3	35	8.9
Total	145	100.0	100	100.0	91	100.2	24	100.0	16	100.2	16	100.2	392	100.0

the curriculums of 'A' and 'B' grades combined, and 'D' and 'E' grades combined. It is obvious here that the best grades are obtained by liberal arts students, 38.0 per cent achieving grades of 'B' or better. Pre-medical students are second (25.0 per cent), engineering third (22.1 per cent), commerce fourth (19.8 per cent), pre-legal fifth (18.8 per cent), and pharmacy last (6.3 per cent).

The combined 'D' and 'E' grades show again that liberal arts students made the best record, 25.0 per cent having received a below-average grade. Next in order they are commerce (29.8 per cent), pre-legal (31.3 per cent), engineering (34.4 per cent), pre-medical (54.1 per cent), and pharmacy (56.3 per cent).

Since, in general, students drop a course because they are in fear of failure, it is interesting to see the number who either failed or dropped the course. The 'mortality' is as follows:

	Per cent
Liberal arts	14.0
Pre-legal	18.8
Engineering	24.1
Pre-medical	25.0
Commerce	34.1
Pharmacy	62.6

Certainly the liberal arts students stand out in achievement in college algebra.

TRIGONOMETRY

Table III shows the distribution of grades in trigonometry in the various curriculums together with the per cents for

TABLE II

Combined 'A' and 'B' and 'D' and 'E' Grades in College Algebra

Final grade	Curriculum												Total	
	Engineering		Liberal arts		Commerce		Pre-medical		Pre-legal		Pharmacy			
	No.	Per cent	No.	Per cent	No.	Per cent	No.	Per cent	No.	Per cent	No.	Per cent	No.	Per cent
A and B	32	22.1	38	38.0	18	19.8	6	25.0	3	18.8	1	6.3	98	25.0
D and E	50	34.4	25	25.0	27	29.8	13	54.1	5	31.3	9	56.3	129	32.9

TABLE III

COMPARISON OF GRADES MADE IN TRIGONOMETRY

Final grade	Curriculum												Total	
	Engineer-ing		Liberal arts		Commerce		Pre-medical		Pre-legal		Pharmacy			
	No.	Per cent	No.	Per cent	No.	Per cent	No.	Per cent	No.	Per cent	No.	Per cent	No.	Per cent
A	10	7.0	9	14.8	6	14.0	2	2.6	—	—	1	8.3	28	8.1
B	22	15.4	13	21.3	4	9.3	9	11.8	2	22.2	—	—	50	14.5
C	48	33.6	26	42.6	17	39.5	23	30.3	5	55.6	4	33.3	123	35.8
D	20	13.9	8	13.1	6	14.0	21	27.6	1	11.1	3	25.0	59	17.2
E	28	19.6	5	8.2	4	9.3	19	25.0	1	11.1	4	33.3	61	17.7
Drop	15	10.5	—	—	6	14.0	2	2.6	—	—	—	—	23	6.9
Total	143	100.0	61	100.0	43	100.0	76	99.9	9	100.0	12	99.9	344	100.2

each. The largest per cent of 'A' grades is found in the liberal arts curriculum, 14.8 per cent, with commerce a close second at 14.0 per cent. Engineering is third (7.0 per cent) among the four large groups. The 8.3 per cent figure in pharmacy cannot be taken as very valuable since so few students are included.

At the other end the lowest number of failures is to be found in the liberal arts curriculum (8.2 per cent) with commerce again second (9.3 per cent). Next in order are the pre-legal (11.1 per cent), engineering (19.6 per cent), pre-medical (25.0 per cent), and pre-pharmacy (33.3 per cent curriculums.

Table IV shows the comparison among the curriculums of 'A' and 'B' grades combined, and 'D' and 'E' grades combined. It is here seen that by far the best grades are attained by students in the liberal arts curriculum, 36.1 per cent making grades of 'B' or better. The next highest record is made by commerce students (23.3 per cent). Then come in order, engineering (22.4 per cent), pre-legal (22.2 per cent), pre-medical (14.4 per cent), and pharmacy (8.3 per cent).

Again the liberal arts students show the best record in combined 'D' and 'E' grades, only 21.3 per cent falling below the average. Next in order are the pre-legal (22.2 per cent), commerce (23.3 per cent), engineering (33.5 per cent), pre-medical (52.6 per cent), and pharmacy (58.3 per cent) curriculums.

TABLE IV

Combined 'A' and 'B' and 'D' and 'E' Grades in Trigonometry

Final grade	Curriculum												Total	
	Engineer-ing		Liberal arts		Commerce		Pre-medical		Pre-legal		Pharmacy			
	No.	Per cent	No.	Per cent	No.	Per cent	No.	Per cent	No.	Per cent	No.	Per cent	No.	Per cent
A and B	32	22.4	22	36.1	10	23.3	11	14.4	2	22.2	1	8.3	78	22.6
D and E	48	33.5	13	21.3	10	23.3	40	52.6	2	22.2	7	58.3	120	34.9

TABLE V

High School Preparation of Students in College Algebra

	Grade	Curriculum					
		Engineering	Liberal arts	Commerce	Pre-medical	Pre-legal	Pharmacy
Average number of semesters in high school	A	7.1	6.0	6.2	7.5	7.5	—
	B	6.7	5.2	5.8	5.0	6.0	7.0
	C	6.8	5.1	5.7	5.2	5.2	5.2
	D	6.3	5.0	5.7	4.8	6.3	—
	E	6.0	4.6	5.3	4.0	4.0	4.9
	Drop	6.1	4.5	4.4	4.0	4.0	5.0

The 'mortality' in the curriculums is as follows:

	Per cent
Liberal arts	8.2
Pre-legal	11.1
Commerce	23.3
Pre-medical	27.6
Engineering	30.1
Pharmacy	33.3

HIGH SCHOOL PREPARATION

The results of an examination of the grades made in the various curriculums compared with the number of semesters of mathematics taken in high school by the students who made those grades are shown in Table V for college algebra and Table VI for trigonometry. Almost on all counts the engineering students were better or as well prepared than were the others. And yet their achievement in college algebra and trigonometry in college was far from best. At the same time the liberal arts students were among the least prepared but made the best records.

Table VII shows the comparison of the various curriculums as to high school preparation and grade point averages made in freshman college mathematics. The liberal arts group has the highest average although the high school training for this group is among the lowest. The pre-legal and commerce groups come next in average, quite lower than the liberal arts, and the engineering is fourth. Engineering students, although they have by far the most high school preparation, rank fourth for all groups.

SUMMARY

The data of this study show that at the Morgan Park Junior College liberal arts students, although they have the least ex-

TABLE VI

High School Preparation of Students in Trigonometry

	Grade	Curriculum					
		Engineering	Liberal arts	Commerce	Pre-medical	Pre-legal	Pharmacy
Average number of semesters in high school	A	7.0	6.4	7.1	5.5	—	7.0
	B	7.1	5.1	6.7	5.9	6.5	—
	C	6.7	5.3	5.8	5.3	4.8	5.2
	D	6.5	5.6	5.3	5.2	8.0	4.3
	E	6.2	5.0	5.7	5.1	6.0	6.2
	Drop	6.1	—	4.1	4.5	—	—

TABLE VII

*Semesters of Preparation in High School and Grade Point Averages**

Curriculum	Algebra		Trigonometry		Grade point average for both groups
	Number of semesters in high school	Grade point average	Number of semesters in high school	Grade point average	
Liberal arts	5.22	3.18	5.46	3.21	3.19
Pre-legal	5.67	2.87	5.67	2.89	2.88
Commerce	5.67	2.75	6.05	3.05	2.85
Engineering	6.59	2.78	6.65	2.73	2.76
Pre-medical	5.14	2.68	5.34	2.38	2.45
Pharmacy	5.13	1.87	5.50	2.25	2.04

* Grade points are as follows: A, 5 points; B, 4; C, 3; D, 2; E, 1. A student with a C average would have 3 grade points.

perience with mathematics, make the best records in college. Commerce and pre-legal students show the next best records although the data for pre-legal students are not quite so accurate, perhaps, as for the others since so few law students elect mathematics. The engineering students, almost all of whom have had at least six semesters of mathematics in high school, rank fourth, better only than the premedical and pharmacy students whose interests lie in a totally different direction.

It would seem that mathematics teachers should have the right to expect that engineering students would do good work in mathematics. One would expect commerce students to do better work in accounting than would engineering students. By the same reasoning one would expect to see engineering students to lead commerce students in mathematics. Yet that is not the case. No doubt many of these engineering students are in the wrong curriculum, and of course they have to change after failing. If they are in the wrong field someplace along the line their guidance has been at fault.

Conferences with engineering students who have done poor work in mathematics have revealed the following as the most frequent reasons for their choice of engineering as a future profession.

1. Their fathers are engineers and want them also to be engineers.
2. They are thrilled by the prospect of wearing high boots, khaki shirts, and ten-gallon hats.
3. They are very handy at mechanical arts and think that is the important factor in successful engineering.
4. They like nothing else and thought engineering might be as good as anything.

Obviously, many students who think they want to be engineers ought to be discouraged from attempting it.

Notice!

Characteristic Differences in Mathematical Traits of Good, Average, and Poor Achievers in Demonstrative Geometry*

By Harry L. Stein
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1. Purpose and Method of Study

This study was an attempt to determine to what extent good, average, and poor achievers in demonstrative geometry differ among themselves in certain basic characteristics believed to be more or less directly related to successful accomplishment in demonstrative geometry. The motive was to gain information which might be useful in the fields of diagnosis, remediation, guidance and possibly prediction of success.

The method of the study involved, first, the administration of a battery of, for the most part, standardized tests, which, it was believed, would reveal the extent to which the selected groups of achievers differed among themselves. The battery included the criterion test, the Cooperative Plane Geometry Test, Form R (3); the Dominion Group Intelligence Test, Advanced Form A (4); the Revised Minnesota Paper Form Board, Series AA (9); the A.C.E. Psychological Examination, 1941 (1), from which use was made of the Linguistic Score, the Quantitative Score, and the Arithmetic Problems test as separate measures; the Schorling-Clark-Potter Test of Arithmetic Computation (10); the Van Wagenen Reading Scales in Science (11); the Sims Socio-Economic Status Inventory; a locally developed test of logical reasoning in non-mathematical situations; the Cooperative Elementary Algebra Test, Form R (2); and the Wrenn Study Habits Inventory (12). Secondly, the results of the battery were analyzed by means of statistical techniques designed to test the significance of differences among and between the various groups. Thirdly, a study was made of sex differences on the various traits analyzed. The analysis of variance and covariance was the major statistical technique used (5, 6, 8).

Before the various tests could be analyzed certain assumptions basic to the techniques utilized had to be satisfied. The population sampled had to be described and the method of sampling indicated. The population sampled was the eleventh grade of the City of Winnipeg high schools and before these schools could be combined it was necessary to test the hypotheses of homogeneity of variance and achievement. When the tests of homogeneity were applied it was found that when all the classes and schools were combined, the assumption of homogeneity of achievement was not satisfied. It was necessary, therefore, to eliminate two classes in order that the assumptions might be satisfied and the schools treated as a single sample. The results of the two classes omitted from the composite analysis were dealt with in a separate section.

The basis for establishing the three achievement groups upon which the analysis of the various traits was made was the Cooperative Plane Geometry Test, Form R. In the composite analysis, 260 students were grouped into approximately upper, middle, and lower thirds, designated as good, average, and poor achievers respectively on the basis of the results of the criterion test. These groups were then compared as to their achievement on the battery of nine tests from which twelve traits were studied.

* This paper is an abstract of the essential features of the author's Ph.D. Thesis at the University of Minnesota.—Editor.

The twelve traits were designated as: 1, general intelligence, 2, spatial relationships, 3, linguistic ability, 4, quantitative ability, 5, total score on the A.C.E. Psychological Examination, 6, arithmetic problem solving ability, 7, arithmetic computational ability, 8, reading comprehension, 9, study habits, 10, logical reasoning ability, 11, symbol manipulation, and 12, teachers' estimate of success.

Finally, an attempt was made to determine which combination of the twelve designated traits would form the best basis for prediction of success in demonstrative geometry. A multiple correlation coefficient was obtained on the basis of five factors which appeared to be most closely related to the criterion. However, when the standard partial regression coefficients of these five traits were computed, only two of them, those based upon the Dominion Intelligence Test and the Cooperative Elementary Algebra test, were found to be significant. A multiple regression equation was therefore derived on the basis of these two factors.

2. Principal Findings

The analysis of the results of the battery of tests disclosed the following findings:

1. The means of the three achievement groups differed significantly at the one per cent level on the basis of the criterion.

2. Significant differences were found to exist at the one per cent level between the means of good and average, good and poor, and average and poor achievers in intelligence as measured by both the Dominion Test and the A.C.E. Psychological Examination, linguistic ability, quantitative ability, symbol manipulation, and teachers' estimates.

3. In spatial relationships, as measured by the Minnesota Paper Form Board, a significant difference existed at the one per cent level between the good and poor achievers, and at the five per cent level between the good and average achievers.

4. In reading comprehension, the difference between the means of good and average achievers was significant at the five per cent level while in study habits. the difference between the good and poor achievers was significant at the five per cent level only.

5. In only two traits were there no significant differences between any pair of groups. Both of these were between the average and poor achievers. The traits were spatial relationships and study habits.

6. It seems evident, then, that significant differences among the means of achievement of all groups appeared to exist in all but two minor cases.

7. On the basis of the criterion boys were significantly better than girls in demonstrative geometry. However, on the basis of teachers' estimates, girls were significantly better than boys.

8. The only other trait in which a significant difference appeared between boys and girls at the one per cent level was in arithmetic computation, in which the mean achievement of the boys was greater than that of the girls.

9. The best single factor predictive of success in demonstrative geometry was intelligence as measured by the Dominion Intelligence Test. The second best factor was elementary algebra of the ninth grade.

10. A multiple correlation coefficient $R_{1.23} = .665$ was obtained for the relationship between the estimated criterion scores and the intelligence and algebra factors.

11. The multiple regression equation for predicting criterion scores X_1^1 from the intelligence and algebra scores X_2 and X_3 was found to be

$$X_1^1 = .346X_2 + .315X_3 - 7.25$$

with a standard error of estimate of 6.837.

12. Analysis of a limited number of cases revealed that the general intelligence factor was the greatest common factor in all the traits studied.

Table I shows the critical ratios obtained from the analyses of variance among and between the groups selected on the basis of achievement on the criterion test.

3. Conclusions and Implications

1. The factors most closely related to success in demonstrative geometry were general intelligence and ability to manipulate symbols as in algebra. Many other mathematical traits such as ability to perceive and comprehend spatial relations, the ability to perform arithmetic computations and to solve arithmetic problems, the ability to read with comprehension, general numerical or quantitative ability and the ability to reason logically have a direct relationship to success in geometry, but the extent to which these traits depend upon general intelligence, and the extent to which they themselves are interrelated makes it difficult to gauge their effect. The multiple regression equation shows that they make no significant direct contribution to prediction.

2. From the results of the test of logical reasoning in non-mathematical situations it is evident that the students in this sample have not developed the ability to apply logical reasoning to situations outside of geometry. These results may be due to the non-validity of the test itself. However it is more probable that either the method by which they were taught does not bring about a transfer to non-mathematical situations, or the students are not sufficiently mature to make the transfer. On the other hand, it may be that the development of the ability to reason logically in all situations as an outcome of demonstrative geometry is purely mythical.

3. Knowledge of specific weaknesses as determined from the test battery may be of considerable value in improving achievement in geometry. If poor linguistic ability or poor computational ability are diagnosed, remedying these weaknesses might improve achievement greatly, since the good achievers evidently possess these specific abilities in strong measure. Modern diagnostic theory emphasizes the necessity of breaking down psychological areas into more specific and therefore more comprehensible factors.

Table I

Summary of Critical Ratios Obtained from Analyses of Variance and "t" Tests of Twelve Traits upon Which Good, Average, and Poor Achievers in Geometry Were Compared

Traits Measured	Good and Average Achievers	Good and Poor Achievers	Average and Poor Achievers
Intelligence	4.788	11.784	5.499
Spatial Relations	2.389[1]	4.039	1.577[2]
Linguistic Ability	2.608	6.004	2.822
Quantitative Ability	2.687	7.208	4.526
A.C.E. Total Score	3.278	7.943	4.191
Arithmetic Problem Solving	4.411	7.331	2.464[1]
Arithmetic Computation	4.195	6.279	2.048[1]
Reading Comprehension	1.988[1]	5.276	3.156
Study Habits	2.610	2.102[1]	.057[2]
Logical Reasoning	2.693	5.894	2.428[1]
Symbol Manipulation	2.777	6.198	4.625
Teachers' Estimates	5.373	10.032	4.557
Geometry (Criterion)	19.804	31.325	19.615

[1] Significant at five per cent, but not at one per cent level.
[2] Not significant at five per cent level.

4. From a guidance point of view the study brings out the need of knowing possibly in advance of the study of geometry at what points the student is likely to have difficulty. If a survey of specific abilities is made early enough it might be possible to prevent poor achievement by bolstering up those abilities in need of attention before they bring about inhibitions, discouragement, and possible failure.

5. The guidance viewpoint should again be considered in the matter of deciding who should or who should not elect a course in geometry. It should be possible, by means of the regression equation, to

predict within limits the likelihood of success or failure. The use of the Johnson-Neyman weighting factor (7) in the formula for the standard error of estimate enables us to set up confidence belts for predicted scores.

6. The need of individual diagnosis is evident from the case studies sampled. Obviously, a factor which may inhibit success in one individual may not be as important a retarding factor in another. Likewise, the possession of a larger measure of ability in any trait or group of traits may not insure success for an individual since some minor factor may be sufficient to cause failure.

7. Many factors other than those considered in this investigation may have a direct bearing upon success or failure. The skill and personality of the teacher may produce success even in the face of evident psychological weaknesses. The will to achieve may be very effective as a spur to success even when innate ability to achieve is limited. The difficulty of measuring characteristics such as these does not mean that they should be neglected in any diagnostic or guidance program.

4. Problems for Further Research

Some of the traits dealt with in this study have appeared to bear a less direct relationship to success in geometry than others. This does not mean that they are unimportant. It may be that more appropriate methods of measuring these traits might reveal a closer relationship to success in geometry. The spatial relations factor, for example, warrants further study. The items of the Minnesota Paper Form Board are all of a similar type. The development of items concerned more directly with the subject matter of geometry would be of real value. Items dealing with overlapping triangles, parallelism, circles and other plane figures should be studied for their validity in a spatial relations test.

Another trait not considered in this study was orientation. The construction of an adequate test of spatial and directional orientation may prove to be of value.

The extent to which memory is a factor in success in geometry was not considered. It might be worth while to study the effect of ability to memorize various types of material, both mathematical and non-mathematical, in relation to success or failure in geometry.

The writer is not satisfied with the outcome of the test of logical reasoning in this study. An attempt should be made to investigate the extent of ability to reason logically both before and after the course in geometry. If the necessary controls are instituted it might be possible to measure more closely the effect of a course in geometry upon the ability to apply logical reasoning to both mathematical and non-mathematical situations.

A study might be made of the types of computational operations needed for demonstrative geometry, together with the specific units of measure used in this branch of mathematics. Information of this type would be valuable in a program of readiness for geometry.

An aspect of the whole situation which was also not given consideration might be referred to as the aesthetic appreciation side of geometry. It may be that a realization of the beauty and symmetry of nature, and of certain types of architectural construction may be a means of awakening an interest which might result in improved achievement. The construction of a measure of appreciation and interest in geometry would be a valuable asset to diagnostic study in this area.

The problem of testing out the regression equation on samples of populations in various areas would be of value. It may be advisable also to test the significance of standard partial regression coefficients based on factors other than those upon which the regression equation in this study is based.

By means of the discriminant function the data could be analyzed to determine what proportion of the sample would likely fall into any particular category such as

the good or the average or the poor group so far as the criterion is concerned.

A factor analysis of the thirteen traits used in this study might be made to determine whether or not primary traits exist. How useful this information would be from a practical standpoint is a matter of conjecture.

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Mathematics Skit

By John Curtis Gowan
Culver Military Academy, Culver, Ind.*

Time: The present. Scene: The office of Mr. Strumpelbein, third vice president of the Consolidated Construction Corporation, who has just risen to greet an inspector from the U. S. Government.

Inspector: Good Morning, Mr. Stumble-in, I'm a special inspector from the luxury tax division of the Bureau of Internal Revenue, and they told me you were the third vice president in charge of planning and so the gentleman I want to see.

Strumpelbein: (Affably) Inspector from the Internal Revenue Department, eh? Well, glad to meet you inspector. (Pointedly) My name is Strumpelbein. I'm sorry, but we have no luxuries here. This is the place where we figure out how to do all the different jobs that the many activities of our corporation require.

I: I'm sorry, Mr. Stumble-bum (here, Mr. Strumpelbein corrects him, and continues to do so when necessary throughout the action), but you're wrong about luxuries. The Supreme Court has just decided that higher mathematics is a luxury, and therefore taxable as such under the recent luxury tax law.

S: (Stupefied) You mean to say we have to pay to use mathematics?

I: Anything higher than arithmetic you do. The Supreme Court said so in the case of .... (insert some appropriate teacher's name) versus the state of .... Want to see the ruling, eh? (Strumpelbein shakes head) Well then, here's the list of assessments (Hands over paper)

S: (Reads haltingly, with mounting indignation): Factoring, 3 dollars and a half a hundred. Quadratics, 4.50 a quad. Binomial theorem, 6.75 an expansion. Logarithms, 50 cents for the first log and 10 cents for each additional log thereafter. Radicals, 80 cents per square root. Calculus, 9.50 a derivative. Probability, 10 cents a chance, three chances for a quarter. Statistics, taxed by the yard. Infinitesimals, tax free.—Why are infinitesimals tax free, inspector?

I: Well you see the tax is also infinitesimal.

S: Oh, (Continues) Permutations and combinations, 12 dollars a ton, h'm, almost as expensive as coal. Arithmetic series, 14 dollars each. Simultaneous quadratic equations, $198.50 (good riddance, never did like them anyway). Percentage problems, 10% of the answer. Work problems, except defense work, 25% of the salary. Mixture problems, 50 cents unless you get the right answer.—What happens, inspector, if you get the right answer?

I: Who does?

S: Oh. (Continues) Simple equations, 3 dollars apiece. Graphs, taxed on a sliding scale according to the slope, and long division, the government gets the quotient, and you get the remainder,—(ANGRILY) why this is the most preposterous thing I ever heard of in my whole life!

I: (Suavely) It may be preposterous Mr. Bumble-stem, but that doesn't mean the government can't do it. In fact they HAVE done it (SHOWS ENORMOUS BADGE) and I'm here to check up on your outfit and to collect on all the mathematics you've been using around this place. (He advances toward Mr. Strumpelbein, who, cowed by this display of power, has cringed down behind his desk.)

S: (Depreciatingly) Why we use very little mathematics in our business, inspector. It's hardly necessary at all.

I: (Still belligerent) Well, we'll see about that, Mr. Rumble-seat. Now let's have a list of your projects for the past year.

* This skit was presented during the Mathematical Assembly at Culver Military Academy, January 26, 1942.

(Gets papers). H'm, what's this first item I see here, "Best shape and size for cans and containers?"

S: Well, we did do a little work figuring out how we could save money by using a better shaped tin can for our product which would make it hold more and at the same time use less tin.

I: Oh you did, eh, and just how did you manage that, by differential calculus, or did you use an ouija board?

S: Well, I'm afraid we couldn't afford to guess on that one, so we used calculus. (Hurriedly) But that was just a lonely example.

I: Not so lonely if my eyes serve me right. Says here that you've been doing some research on the cooling and annealing of metals and glass. (SARCASTICALLY) I suppose that didn't take any differential equations?

S: Well that was pretty technical, but now our usual line of work—

I: (INTERRUPTS) Uses mathematics even more often. How about this report that your big machine tools were turning out 8% defective parts?

S: Well, we did use some statistics there.

I: Yeah, and you used some more in weighing variables for these index numbers I see in this cost study.

S: Yes, but that just comes once a year.

I: So do taxes, Mr. Stroople-bean. I guess I came to the right place all right. Now how about these blueprints. Here's one involving factoring and quadratic equations, all in the same wheelbase problem.

S: What a spendthrift our draftsman must be!

I: And look at this report on this new alloy you're making, 10% iron, 15% phosphate, 50% alcohol, 12% nicotine, and 3% Alka-seltzer! What the devil is that,—a rat poison, a new vitamin, rayon for girls' stockings, or a pick-me-up for the morning after?

S: (AT LAST A TRIUMPH) I can't tell you. It's a military secret!

I: Well it's no secret it will cost you a pretty penny for all the mixture problems and formulas, I see there. Now I suppose you amortize your depreciation, by establishing a sinking fund, and every year charging off a certain portion. (S NODS) Well, that's mathematics of finance. And do you have an insurance plan for employees? (ANOTHER NOD) Well, that lets you in for the tax on probability. Now for a quick look around the office. Say, this place is lousy with mathematics! What's this chart over here?

S: Now you can't jump on us for that. That's just a plain chart of our past growth and estimated future expansion.

I: Well, expansions come under series, so it may be a plain chart to you, but it's ten dollars to us. Well, here's your bill, Mr. Stinckel-boom, two hundred and fifty five dollars and forty seven cents!

S: Gad! but you can't do this to us! We'll fight it tooth and nail. Suppose we won't pay?

I: Then you'll have to stop using mathematics in your business, Good day Mr. Bumble-bean! (EXIT)

S: (MAD CLEAN THROUGH) But we can't stop using mathematics in our business, we or anybody else. Mathematics is at the very foundation of business,—its a prime necessity, I tell you, not a luxury, a necessity. (HE IS STILL HOLLERING OUT THE DOOR AT THE DEPARTED INSPECTOR WHEN SUDDENLY THE IMPORT OF HIS WORDS STRIKES HOME) By George It *IS* a *NECESSITY!!* THAT'S what we'll tell the Court! That's what we'll tell the Congress! That's what we'll tell the world!! (HE STRIDES DECISIVELY BACK TO HIS DESK) Gimme that phone! Gimme Washington!! Gimme the Supreme Court!!! GIMME FRANKFURTER!!!!——(BUT THE CURTAIN HAS CLOSED.)

Mathematics in Relation to Curriculum Adaptations, Pre-Induction Courses, and the Victory Corps*

General Policy

It is recognized that the abilities of high school boys and girls range from that which is limited to the mastery of simple arithmetic, intuitive geometry, and the use of simple formulas, to the ability which permits the complete understanding of advanced abstract principles of algebra and geometry.

It is recognized that the army, navy, industrial, and professional organizations in their various branches have need for applications of mathematics which run through the same range of required abilities. To meet these needs immediately the following guiding principles are proposed:

1. With few exceptions boys in the high schools should study mathematics, either the regular academic mathematics or general mathematics or special courses in mathematics.

2. Superior students in mathematics should be advised to take four years of high school mathematics including trigonometry and solid geometry. For those schools which find it possible additional work in advanced algebra, spherical trigonometry, analytic geometry, and calculus will be beneficial.

3. For the boys who will stay in high school more than a year it is not necessary to create new courses or to entirely reorganize the present courses. What is needed is a shift in emphasis to the practical phases of the subject.

4. For the boys who will graduate in June 1943, or who will become eligible for the draft within a year, and who are not now studying mathematics create an abbreviated course. If the boys have had one or two years of high school mathematics, this course may be a refresher course in the mathematics they have had, along with a review of arithmetic and arithmetic reasoning. Since such a course gives new emphasis, it should be accredited towards the high school diploma in addition to previous credits in mathematics. For those boys who have had no high school mathematics give a course in general mathematics that contains the elements of arithmetic, scale drawing, intuitive geometry, numerical trigonometry, and simple algebra.

5. The subject matter and the rate of teaching must be modified so that complete understanding and accuracy and skill in application are attained. This is the prime requisite. Even the best modern texts contain a considerable amount of material that can be safely eliminated. If too many topics are covered, the pupils

* Members of the Council who attended the Bethlehem meeting in December 1942 will remember the report of the study of the Defense Committee of the National Council of Teachers of Mathematics made by its chairman, Professor Shuster.

We have just received a bulletin of the Curriculum Advisory Committee of New Jersey in which practically the same conclusions as those reached by our committee have been restated in such a clear and helpful manner that we feel they should be given the publicity they deserve by printing them in the Teacher.

The members of the New Jersey Committee are:

H. H. Ryan, State Teachers College, Montclair

H. F. Fehr, State Teachers College, Montclair

Carl N. Shuster, State Teachers College, Trenton

Earl R. Glenn, State Teachers College, Montclair

Lehman C. Shugart, Supervisor of Science, Elizabeth

Paul Spencer, Central High School, Trenton

C. H. Threlkeld, Columbia High School, South Orange

G. Hobart Brown, Roselle Park High School

H. A. Titcomb, Neptune High School, Ocean Grove

Charles W. Hamilton, State Department of Public Instruction

Howard Dare White, State Department of Public Instruction

—The Editor.

will not receive sufficient drill to secure mastery of the important phases of the subject.

6. The mathematics now taught in high school should be practical to the extent that it has immediate application or that it is needed for other essential mathematics or that it pertains directly to the war effort. This means the mathematics teachers must have at least a fundamental understanding of the vocabulary and elementary problems in marine and air navigation, artillery fire, aeronautics, cartography, physics, and shop work. This knowledge can be gained by self study of the existing literature in these fields or by taking courses in these fields offered by the colleges and universities.

7. Girls who have demonstrated ability in mathematics must now be encouraged to study four years of high school mathematics. These girls, after an accelerated college training, will be needed (the need is even now acute) to fill the vacancies in scientific laboratories, specialized industrial work, and the teaching profession. There is an immediate need for high school girls with such training in the signal corps and in certain specialized industrial work.

8. The regular high school mathematics courses, well taught, are essential in preparation for the war effort. Courses in aeronautics or navigation should not be substituted for courses in basic mathematics and should be offered only when there is available a teacher who has had practical experiences in aviation, navigation, field engineering, artillery fire, etc., and when adequate equipment is available.

9. The basic adjustments suggested are intended to prepare pupils to function efficiently in the whole war effort. However, the revisions suggested will lead to better work in the mathematics that will be needed after the war has been won.

Specific Suggestions

Changes in organization, content, and methods of teaching, require careful planning. The immediate needs for war service may be met by the elimination of certain less important material and the substitution of more practical topics. Experiences of the war needs suggest the following specific recommendations.

1. A refresher course, besides reviewing the fundamental principles of algebra and geometry, should include a thorough treatment of arithmetic computation, arithmetic reasoning, intelligent use of simple principles of approximate computation, formulas, simple equations, graphs, informal geometry, scale drawing, numerical trigonometry including the law of sines, the slide rule, and the use of the artillery mil.

2. For those students who have had no mathematics give a general mathematics course. The emphasis should be on arithmetic computation and reasoning, stressing fractions, decimals, proportion, and per cent. There should also be included a simple practical treatment of formulas, graphs, scale drawing, and numerical trigonometry of the right triangle confined to the sine, cosine, and tangent functions.

3. The following topics should be eliminated or greatly reduced in either a refresher or general course; special products and factoring excepting the difference of two squares and the perfect trinomial square; fractions with other than monomial denominators, and all complex fractions (the type of fractions retained should be those found in formulas in geometry, physics, aeronautics, and simple life situations); equations containing artificial fractions; complex work in radicals (radicals retained should be those occurring in geometry, numerical trigonometry, and physics); quadratic equations other than the type $x^2=k$; systematic deductive logic in the establishing of geometric relations (the treatment should be informal and practical).

4. In the eleventh and twelfth years of academic mathematics diagnostic tests

should be administered to determine any weaknesses in arithmetic computation and reasoning. Remedial work should be given to insure skill and accuracy in computation and ability in reasoning on the part of all students of regular academic mathematics.

5. In the teaching of plane geometry include sufficient work in arithmetic, algebra, and numerical trigonometry to retain and extend skills previously acquired. Where possible, practical field work in which the angle mirror, plane table, hypsometer, compass, clinometer, sextant, and (or) transit are used will add greatly to the value of geometry. For the present, application of deductive reasoning to nongeometric situations should be eliminated.

Bibliography for Teachers

The following annotated bibliography is illustrative of books in the various categories. For a more extensive list see the Bulletin of the Association of Mathematics Teachers of New Jersey, October, 1942, to be secured from Mary C. Rogers, 425 Baker Avenue, Westfield, New Jersey.

Applications to Air Navigation

"Mathematics in Aviation" by George Osteyee. The Macmillan Company, New York, 1942. xii+186 pp. $0.64. Air Age Education Series.

There are 22 chapters dealing with principles of mathematics and mathematical problems as related to aviation. The introduction provides definitions and terms used in air navigation. Applications are in arithmetic, algebra, geometry, and trigonometry. There are many photographs, diagrams, a selected bibliography, and list of pertinent films.

"Practical Air Navigation" by Thornton C. Lyon. Bulletin No. 24, United States Government Printing Office. $1.00.

This is a pre-flight course covering the topics of maps, charts, the compass, air instruments, avigation, radio navigation, and celestial navigation. Many diagrams and simple explanations will enable the teacher to quickly gain a general knowledge of aeronautics.

"Navigation and Nautical Astronomy" by B. Dutton. Seventh Ed. United States Naval Institute, Annapolis, Maryland, 1942. vii +509 pp. $3.00. Mathematical Supplement by Dodson and Hygatt, 32 pp., $0.05.

This is the official text used in the United States Naval Academy. It includes terms, definitions, the compasses, sailings, maps, piloting, radio, nautical astronomy, celestial observations, sextant, determination of position, and aerial navigation. Every type of problem is illustrated by a complete solution. The diagrams are numerous, large, and well drawn. All formulas are derived in the text. The supplement gives a complete review of all high school mathematics.

A Refresher Course

"Basic Mathematics" by Walter W. Hart. D. C. Heath and Company, Boston. vi+456 pp. $1.52. Brief Edition, $1.20.

A survey of secondary mathematics covering the fields of arithmetic computation, elementary geometry, elementary algebra, mensuration, the slide rule, logarithms, and numerical trigonometry. The complete edition includes in addition work in solid geometry, demonstrative geometry, and advanced algebra. Many practical problems pertain to military and naval use. The book contains all the mathematics necessary for pre-induction training.

"Basic Mathematics" by William Betz. Ginn & Company, New York. x+502 pp. $1.48. Contains the same work as reviewed above.

NOTE: Basic mathematics and refresher courses should be used only in case students have had previous high school mathematics. Students without such training should use a general mathematics text including scale drawing and numerical trigonometry.

Military Applications

"Some Military Applications of Elementary Mathematics." Institute of Military Studies, University of Chicago, $0.15.

A complete solution of twenty applications to real military problems involving arithmetic, algebra, geometry, and trigonometry. An excellent guide for application in pre-induction mathematics.

"Elements of Map Projection" by C. H. Deetz and O. S. Adams. Special publication no. 68. Coast and Geodetic Survey. United States Government Printing Office. 200 pp. $1.00.

An elementary presentation of the principles of map projections. Practically all types of projections are described, including the Mercator, azimuthal, gnomonic, polyconic, and the grid system used in the United States Army. There are 53 figures and diagrams.

"Globes, Maps, and Skyways," by Hubert A. Bauer. The Macmillan Company, New York, 1942. viii+75 pages. $0.48. Air Age Education Series.

"Orientation for the Coast Artillery Battery Officer." The Coast Artillery School, Fort Monroe, Virginia, 1942. $0.75.

An official artillery manual used in the officers schools. It covers the practical applications of map projections and grid coordinates to setting up guns, the use of the instruments, the transit traverse, intersection, resection, and azimuth determination. The examples are given in military forms and solved in all detail.

"Elementary Mathematics in Artillery Fire," by Joseph M. Thomas. McGraw-Hill Book Company, New York, 1942. $2.50.

The committee recommends that the teacher of mathematics, if he has not already done so, should acquaint himself with the material of a book in each of the above categories. In the immediate teaching of mathematics as far as possible, the following topics should be thoroughly covered: the use of numerical tables; the slide rule (including the circular slide rule which is used by the armed forces); logarithms; arithmetic computation; approximate computation; graphic solution; reading graphs; numerical trigonometry including scalene triangles; the mil system of angle measure; scale drawing; charts and maps; formulas containing letters with subscripts and letters other than x and y; the ability to read a vernier; and the use of the micrometer, compass, plane table, transit, and sextant.

Important Mathematics Conference

The Rocky Mountain Section of The National Council of Teachers of Mathematics will join with the Mathematics Section of the Eastern Division of the Colorado Education Association and the Rocky Mountain Section of the Mathematical Association of America in sponsoring a conference on the teaching of mathematics to be held in Denver April 16 and 17, 1943. The theme of the conference will be *Mathematics in War and Peace*.

The meetings will be on the campus of the University of Denver. Friday evening a fellowship banquet will be served for all three organizations and other interested friends, followed by a lecture by a guest speaker. Saturday morning two section meetings are planned, one for elementary and junior high school teachers and the other for senior high school and college teachers. After the section meetings the guest speaker will present a concluding message.

The joint committee members who have made the plans for the conference state that the reservations for the banquet must be sent to the secretary, Mary C. Doremus, 1416 Pennsylvania Street, Denver, by Wednesday, April 14. Because of the present difficulties in making dinner arrangements, only those who have made reservations will be permitted to attend the banquet.

The Effect of the War on College Women and Mathematics*

By Julia Wells Bower
Connecticut College, New London, Conn.

It is my privilege to consider with you the effect of the war on women and mathematics, particularly in the women's colleges.

Our students are affected by the restlessness which is evident in their generation outside academic halls. They are faced with problems inherent in their situation: shall I continue my college education, or shall I take an immediate war job? shall I accelerate my graduation? shall I marry my fiance now or wait until vacation? shall I interrupt my college work to spend a last week with my husband before he may be sent abroad? It is a tribute to their undamental seriousness and good judgment that their college work has been disrupted so little.

Let us look now at the problems which beset the institutions that are training them. Immediately after Pearl Harbor the great American shortage in scientific personnel became evident, especially in fields which, like mathematics, require considerable preliminary training. Colleges quickly considered in what way this lack might be made up. Many were the announcements of schemes hastily concocted as well as of programs carefully considered. As time went on, the Army and Navy presented more detailed schedules of their requirements. When the eighteen-year-old draft became a reality, colleges whose student population was in part or wholly masculine set about providing for the specific training which they would be expected to give. It was only natural that under duress of their prescribed and technical curriculum, they should look to the women's colleges to carry on the traditional liberal arts training which they were having to abandon. Solemn indeed were the pronouncements that if American classical culture was to be preserved, it must be in the women's liberal arts college. This is all too true, for there is danger that at least one whole college generation will have received training only in the practical.

Scarcely had the statement been made, however, when it became apparent that if men were to be drawn in large numbers from scientific research or from technical jobs, then women would have to replace them. This would mean that women must be urged to go into scientific studies to the neglect of the humanities. For, unfortunately, the student who is equipped with powers of concentration and organization sufficient to fit her to do the classical subjects exceptionally well, and thus be one of those who would preserve the classical tradition, is also a student who can do scientific work well even though she may not be so happy in doing it. And so women's liberal arts colleges are faced with the dilemma of meeting war demands and at the same time carrying on in the fields which will be so much needed during the reconstruction. When you consider that the longer a nation is at war, the more liberal education tends to be superseded by technological, and that education in the occupied countries is confined to the most elementary aspect of the three R's, you can understand the seriousness of the need for well trained latinists as well as well trained physicists.

Since the war need is immediate and pressing, we have, of course, tended to try to meet it, hoping that by some miracle we will not be caught too shorthanded when the very different reconstruction need arises. It is interesting to note that in one

* Paper read at the meeting of The Association of Teachers of Mathematics in New England in Providence, R. I., on March 6, 1943.

of the latest bulletins of the American Council on Education [1] it is emphatically stated that the basic curriculum of a good college of liberal arts or science is essential. The reason given, however, is the very practical one that such training provides essential workers on high educational levels. For example, in fields such as mathematics, graduating majors are useful without further training as well as having the foundation for urgently needed professions such as engineering and teaching. The bulletin notes that many colleges supplement their traditional curriculum with such training as secretarial. Colleges are urged to accelerate the graduation of good students in order that their qualities of mind, character, adaptability and citizenship may be more immediately available. For the exceptional student, this acceleration should impair neither health nor the quality of education. In similar vein, a recent suggestion of the National Research Council urges that we recruit into mathematics and physics all students who have the necessary aptitude [2].

Let us look at a small sampling of the war work which requires the qualifications possessed by a college graduate. If a student likes arithmetic and gets real pleasure from working with figures, then she can look forward to statistical work. If this is done in government service, then her duties will be to collect, edit, and analyze statistical information, to plan and supervise projects, to interpret results, to prepare reports and so on [3]. The responsibility and interest of her assignments will, of course, depend on her training and fitness. Usually, at least one year of experience after the college degree is necessary. If the statistical work is done in a psychological clinic, then the duties and requirements are those peculiar to the clinic in which she will work.

If the student prefers algebra, then she might be interested in going into actuarial work. As you know, in former times women were admitted to this profession only grudgingly. Now, however, we receive requests for recommendations of qualified seniors. Although this is not directly war work, still it is one of the fields left vacant by the drafting of the men and it requires very special training.

If the student takes physics or chemistry with her mathematics, then she is eligible for any of the multitude of engineering-type jobs of which we have heard so much. If she could combine with this six hours of engineering, or take an ESMWT course, she could be a junior engineer with the government, testing and inspecting engineering materials, assisting in experimental research, drawing up plans for minor projects, preparing reports, making computations, and so on [4]. Or she would be eligible for specific engineering training for a private industrial company, such as that given at Yale University for the Vought-Sikorsky corporation [5]. If she does not wish to finish college, but has had two good years, then she could become a technical assistant for the government doing routine work in the engineering field [4]. Or she could train for similar work in a private corporation, for instance become an engineering cadette with Curtiss-Wright and receive at their expense ten months of pertinent training in some school of engineering [6].

On the other hand, if the student prefers theoretical mathematics and is well grounded in the calculus and differential equations, then she could become an assistant computer in government service at Langley Field or on the Aberdeen proving grounds. She would make calculations using trigonometric functions and logarithms in connection with experimental data, would perform mathematical transformations involving a knowledge of the calculus, theory of equations and differential equations, and so on [7]. These jobs are among the most interesting open to college mathematics majors. A close rival are those in meteorology, for which additional training must be taken. Since there must be a trained meteorologist for every so many airplanes sent out, it is

obvious that the supply must be much increased. There are many non-combatant locations in which women can serve. Government scholarships are available to outstanding students who wish to prepare for this work [8]. Here again, in both of these fields if the student were forced to leave college early, she could become a junior assistant, performing such of the above duties as her training would permit.

The traditional field of secondary school teaching is, of course, open to the college mathematics major. I do not need to tell you how much she is needed in this work which has become truly war work with its object of giving sound mathematical training to boys and girls who will shortly be using it in industry or the armed forces.

Let us now see how colleges are adjusting to the demands of this "mathematicians war" as it has been called. A recent study made for the Mathematical Association of America by Professor G. B. Price of the University of Kansas and reported in the January "American Mathematical Monthly" [9] is the source of some interesting information on this problem. A large increase in enrollments in mathematics courses was observed, usually thirty per cent for men's colleges but reaching as high as three hundred per cent, and from twenty-five to forty per cent in women's colleges. At my own school, Connecticut College, we added a third section of freshman mathematics in September. Later in the year we organized still a fourth, primarily for seniors who wished to have the magical "one year of mathematics" before receiving their degree. They realized that their usefulness in the war effort would be increased by this training. These students have valiantly survived one extra class meeting a week and are much to be congratulated on their performance in spite of the long gap of three or more years in their mathematical experience. The actual number of women studying mathematics is still woefully small, even with this large per cent increase.

At the same time that enrollments have been increasing, the number of instructors has, of course, been decreasing, due to the demands of research, industry, and the armed services. Colleges have met this situation by increasing the load of the remaining faculty, by the use of graduate students or exceptional seniors as assistants and readers and by calling back retired professors and other mathematically trained people who have been engaged in other professions. There has been in some schools a very wise attempt to borrow instructors from other departments. Usually these new teachers have taken refresher or advanced courses in preparation for the mathematics teaching.

There has also been the inevitable adjustment by dropping all but the most essential courses. As a result, many theoretical mathematical courses have had to be cut out together with special courses that were sometimes called "frills." There has been a trend back toward standard courses and a new trend toward teaching and research in applied mathematics. At Wellesley College a full year course in radio and electronics has been introduced for students who have had calculus and sufficient work in electricity [10]. At Smith College, war minors have been inaugurated. These consist of electives in applied fields such as aerodynamics, electronics, etc., for those who have the necessary qualifications [11]. Vassar College has gone a little further and established a pre-engineering curriculum which can be taken as a major in mathematics or physics [12]. It consists basically of mathematics through differential equations, two years of physics, including advanced electricity, one year of chemistry, one year of mechanical drawing, blueprint reading, etc., one semester of mechanics, and one year of economics. Electives are in the field of mathematics, physics, foreign language, astronomy, geology, etc.

There has been also a tendency, encouraged by the U. S. Office of Education [13] for institutions to give special college

training in short courses or special curricula. Some are designed particularly for women. For instance, at Purdue, there is training for statisticians and at the University of Chicago for women in electronics. At the University of Pennsylvania there is a course for women in engineering drafting which requires considerable mathematics. The University of California at Los Angeles gives a special course for aircraft workers. In the summer school of Smith College mathematics and science will be emphasized. At Connecticut College we will give a special six-weeks summer course for a Connecticut industry, training college graduates to their specific requirements.

All this means, of course, that advanced theoretical mathematical instruction in undergraduate college or graduate school has had to be sharply curtailed. It is here that we in the profession are faced with the dilemma that I mentioned before: that of supplying the needed workers to meet the present war crisis, and at the same time providing a sufficient number of trained young people to carry on research when the war clouds are lifted. We, too, tend to provide for the immediate need hoping that by a miracle we will also be able to meet the future need when it arises. We are fortunate in having good reason to think that this miracle may come to pass. It is good sound mathematical training that is being needed and used. Hence, the basic knowledge will be present when our graduate schools are again equipped to build a research structure upon it.

Women in mathematics are indeed fortunate in that they can major in a subject which they find challenging and stimulating, but which also gives them training for work that is desperately needed and that can be done only by those having that training. They have the rare privilege of working in a field which they thoroughly enjoy and at the same time of making their proper contribution to the war effort of their country and to the reconstruction that is to come.

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Notice!

The summer (1943) meeting of the N.E.A. has been cancelled. This means that there will be no summer meeting of the National Council of Teachers of Mathematics this year.—Editor.

Some Suggestions as to the Non-technical Preparation of Teachers of Mathematics*

By Arnold Dresden
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It is at the same time a source of weakness and a source of strength for mathematics as a high school subject that its content is farther removed than most other subjects from the every day experiences of the pupils who are being initiated into its mysteries. The danger arising from this isolation is that the significance, meaning and value of the subject are likely to be seriously underestimated both by the students and by the general public, on whose support the continuance of the subject in the school curriculum ultimately depends. Mathematics is "hard" for some students, largely because it deals with unfamiliar material and because it raises questions, which seem both strange and unimportant to the mind of the boy or girl who has to think of earning a livelihood as well as of passing the course. Why he should learn to solve a quadratic equation or to sum a geometrical progression may seem as puzzling to him as it seems silly to worry about proving that all right angles are equal. Parents are likely to wonder how their children's skill in solving problems about clocks and water tanks will strengthen their character, or how the ability to learn by heart proofs of geometric theorems will stimulate their imagination. That the high school student has not established a connection between mathematics and the rest of his experiences may be natural and no cause for wonder. That the parents, who have "gone through" the schools and possibly a college, have not done so constitutes a serious criticism of our work; it is an indication of the fact that the problem we are here discussing has not found a solution adequate for the needs of the past generation.

* Paper read at the Christmas meeting of the National Council of Teachers of Mathematics at Bethlehem, Pa. on December 31, 1941.

Yet, in spite of these dangers, it is in its very remoteness from daily experience that the unique value of mathematics lies. By dealing with questions in an abstract formulation, it centers attention upon their relevant elements ignoring the accidental ones. What is essential about the fact that $38\cdot 42=40^2-2^2$ is not that 38 and 42 are even numbers, nor that both are two-digit numbers, but that they are respectively the difference and the sum of 2 complex numbers and that they fit into the formula $(a-b)(a+b)=a^2-b^2$, which can be shown to be valid for complex numbers a and b. It is only when the relation of the special case to the general formula has been fully understood, when the particular arithmetic statement has been recognized as the special case of the formula, and the formula as the embodiment of its special instances, that the formula begins to acquire its true significance. The mathematical treatment of problems makes it possible to get at the heart of situations, to discover the essential equivalence of questions superficially distinct from one another, besides providing labor-saving methods. The teaching of algebra, of geometry and of trigonometry furnishes valuable opportunity to develop such insight. To illustrate by a simple example: When a child has \$1 in his savings bank and then deposits in successive weeks 15 cents, 20 cents, 8 cents, etc., he will have, after these deposits, the successive totals \$1.15, \$1.35, \$1.43, etc. If he checks up at the end of some weeks, he concludes that he has deposited $(15+20+8)$ cents and hence that he should have $(1+0.43)$ dollars. A number of such experiences should prepare him for the algebraic statement that $[(a+b)+c]+d\cdots=a+(b+c+d\cdots)$. In a similar way he could be led to understand the

meaning of the formula $[(a-b)-c]-d \cdots = a-(b+c+d+\cdots)$ as well as of other expressions involving "removal or insertion of parentheses." Algebraic knowledge built on a variety of such experiences is likely to penetrate further than the mere learning of rules concerning parentheses, or than the mere playing with bank account problems. But, surely, before an audience of teachers of mathematics it is not necessary to insist on this any further.

All this has been said many times before; both the dangers and the advantages of mathematics have been recognized. Nevertheless, we have not succeeded in elaborating a satisfactory solution of the dilemma thus presented. There have been those who have sought to avoid the danger by "enriching the curriculum"; that is, by centering the attention of the pupils upon so-called life situations. There have been others, certainly fewer in number, who have insisted upon the maintenance of the abstract character, upon the disciplinary value of the subject, upon its formal aspects. In many cases, the introduction of "enriched content" in one part of a course has not prevented extreme formalism in another part. It is symptomatic of this state of affairs that one rarely finds a student who can find the square of 105 without paper and pencil, although he may readily respond with a^2+b^2+2ab to the stimulus $(a+b)^2$. Mathematics deals with abstractions, but they are abstractions from something. Teaching that limits itself to "life situations" may lay a valuable foundation for instruction in mathematics, but it does not in itself provide such instruction; it is important even in the earlier stages that the mathematical content should be extracted without much delay from the students' experience. Instruction, on the other hand, which deals too exclusively with formal aspects, is building on sand. Knowledge reared on formulas alone is scattered by the winds of time during one summer vacation.

It is a recognition of the seriousness of this problem which I want to urge on those concerned with the teaching of mathematics. I believe that the road towards a solution must be sought by an enrichment of the teacher rather than by mere enrichment of the formal curriculum. The teacher should be equipped to draw the mathematical content from a variety of experiences, and to supply the pupils with experiences which will lend themselves to eliciting results of mathematical significance. With such preparation, a resourceful person may be expected to obtain better results from traditional material than can be obtained from an "enriched curriculum" which omits the mathematical implications of the experience, and more lasting insight than from formal instruction divorced from experience. Give the teacher a large supply of material from which to select from year to year, and from class to class, matters best suited to the needs of the particular group that he has to deal with, give him insight into the mathematical content of his pupils' daily experiences, and make him realize that it is his task to provide his pupils with significant experiences and to draw from these experiences such mathematical value as they can produce.

This is a statement of the problem which I am proposing for consideration to those responsible for the training of teachers of mathematics. You will not expect me to give its solution at this time; all I can do is to indicate some general principles which, in my judgment, will have to guide those seeking its solution. Before turning to a presentation of these principles I want to observe that the same problem confronts teachers of mathematics in colleges and in graduate schools even though the particular form which it assumes may vary considerably. Each stage of abstraction rests upon a basis of facts less abstract. For example, the concept of natural number arises from counting, that of integer is based upon counting and measuring, the real number is developed from the rational number, the functional from the function, and the integral from the sum through the various stages of mathematical development. Curiously enough this problem

seems to have been solved on the whole rather better in the field of college mathematics and in the more advanced courses than in the elementary parts of our subject. Is this perhaps due to the fact that the "concrete material" which underlies the abstractions with which he has to deal is more familiar to the teacher of the advanced subjects than the corresponding material is to the teacher of the more elementary fields? It is a fact that practically every course on the calculus develops the fundamental ideas of the subject from matters already familiar to the student through previous experience. A similar procedure is likely to be effective in coping with the inability to "get mathematics" so frequently met among intelligent people.

The first remark I should like to make is that the solution of our problem can never be considered as final. As developments take place in the various fields of science, more and more material may arise which is of importance to the teacher of mathematics from the point of view which I have presented. It is, therefore, a prime necessity for a teacher of mathematics to be informed concerning the work in as many fields of science as possible.

The second observation is that, in our everyday experience, we should keep our minds, eyes and ears wide open in order to detect all elements of mathematical significance which it may offer. I cannot think of a better illustration of this principle than one which I have mentioned many times before and which I am therefore repeating with some apology, namely the following: conflicts between individuals and between groups of individuals, such as nations or labor unions, or parties to lawsuits, are invariably exacerbated by the inability of either side of the conflict to see the point of view of the other side. Mathematicians have long understood that a relation between two variables x and y can be looked at in the form $y=f(x)$ or in the form $x=g(y)$ or even in the form $F(x, y)=0$, and that the relation has not been fully understood until it has been studied from each of these points of view. It should not be necessary to remind an audience of mathematicians of the fruitfulness of this point of view and of its importance throughout our subject (addition-subtraction, multiplication-division, differentiation-integration, and the like). To recognize that insistence upon all aspects of a controversy has its mathematical counterpart in the inversion of relations and operations, should be useful in connecting abstract procedures with more immediate experiences.

A third suggestion is that the task which I am imposing upon teachers calls for cooperative work. It will have become clear to those of you who have followed my argument up to this point that the program I have outlined can be expected to be effective only when teachers have become thoroughly familiar with the basic facts of the physical and the social sciences. It should therefore be the aim of those who are responsible for the training of teachers of mathematics to initiate them in the fundamental principles of at least one field in the physical sciences, at least one in the biological sciences, and at least one in the social sciences. This training should be of such character as to lead the future teacher to continue his interest and become a "semi-expert" in at least one of the three domains. Teachers could then form themselves into groups, wherever possible, including one or more semi-expert from each of the various fields. By interchange of information, they could make available to all members of the group the results of their studies directed to connecting their field and mathematics. Discussion and, ultimately, publication of material of importance for the solution of our problem, growing out of such studies could be expected to help in "enriching" the teachers as well as the curriculum.

Let me close by mentioning a few isolated instances to illustrate what I have in mind. Instead of treating algebra as a formal discipline isolated from everything else, it should be linked up to the fullest extent possible with the student's earlier experience with arithmetic. Lead him to

consider, for example, the calculation of $76\times35+24\times37$. It could be suggested that, instead of determining each product separately, the student devise schemes like the following: $76\cdot35+24\cdot37=76\cdot35+24(35+2)=(76+24)\ 35+24\cdot2=3548$. After having worked through a number of such instances, he may develop an appreciation of factoring as an aid to numerical computation. He may recognize that a transformation like $ab+c(b+e)=(a+c)\ b+ce$, has value, not merely as a formal drill exercise, but as something that can be used in numerical work. Laws of exponents, and factoring as means to facilitate multiplication have a value quite distinct from their questionable "mental discipline" value. Students should be led to appreciate calculations like $27\cdot48=3^3\cdot2^4\cdot3=3^4\cdot2^4=6^4=36^2=1296$ and should be encouraged to use tables of squares, and square roots. Suggestions in this general direction are made in Miss Hendrix's article in THE MATHEMATICS TEACHER for February 1941, with reference to the preparation of the teacher of geometry.

All students of high school age have experienced accelerated motion as well as motion with uniform velocity. Every child has seen the velocity of objects diminish as they rise above the earth and increase as they fall towards the earth. Motion of trains, of automobiles, of bicycles have long been a part of the experience of high school students. From such experiences much valuable matehmatical subject matter can be obtained. We present one example out of many. *A train makes a 100 mile trip from A to B with an average speed of 30 miles per hour, the return trip at 40 miles per hour; what is the average speed for the entire journey*? Many situations of this type lie within the experience of young students. They give rise to the study of harmonic progressions and of the harmonic mean, which in turn points to the necessity of understanding of and skill in the rational operations with common fractions. With such objectives in mind, the train problem can be followed by questions like the following: *A dealer spends \$1000 on tires at \$5 a piece and also \$1000 on tires at \$4 a piece. He throws the two lots of tires together and wants to sell them with a profit of 20%, what should be the price per tire*? Again, by problems whose general form is the following: *A manufacturing plant consists of a machine A_1 costing C_1 and having a period of service of s_1 years, and a second machine A_2 costing C_2 and having a period of service of s_2 years. What is the period of service of this simple plant on the basis of straight line depreciation*? All such problems are instances of the simple theorem that, if (x_1, y_1), (x_2, y_2), $\cdots$ (x_n, y_n), are pairs of values which satisfy the equation $xy=$constant and if $x_1, x_2, \cdots, x_n$ form an arithmetical progression, then $y_1, y_2, \cdots, y_n$ form a harmonic progression and conversely.

The first objective in a program for the non-technical training of teachers of mathematics might well be a systematic survey of the various fields of experience of high school students for the purpose of obtaining material from which mathematical content can be abstracted. This would then have to be analyzed and organized with a view to developing from it the mathematical principles appropriate to the students' stage of development. The principles so obtained could then be used, on the one hand for applications in new problems suggested by the student's experience, and on the other hand, on an experimental basis for the development of further mathematical theories. This is the problem to which the attention of teachers and of teachers' teachers is directed. If it is solved satisfactorily, mathematics is bound to play a role of increasing importance in the education of new generations, in peace-time as well as in war, because it will then lead them to the insight which is essential to the solution of the many problems which beset mankind. Failing such a solution, our subject is probably destined to atrophy as far as the great mass of our people is concerned, and to remain the tool—subject for engineering and other applied sciences, besides being the life work of the born mathematicians upon whose inspired work the development of the subject depends.

Adams Did It 125 Years Ago

Salient Features of the Scholar's Arithmetic

By Ben A. Sueltz
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The Scholar's Arithmetic by Daniel Adams which was printed by John Prentiss of Keene, New Hampshire in 1819 differs from other books of the period and from other editions of Adams. In this volume, Adams used the workbook idea, he featured meaning and understanding, he used tables, and included a section of less commonly used topics which might serve as reference materials for "men in particular callings and pursuits of life." Furthermore, in his advice to scholars, Adams anticipated many of the weaknesses in arithmetic that high school and college teachers now find so apparent. These special features are some which many of us have considered as developments in the twentieth century.

In printing the book, Prentiss set the abstract examples in such a way that the work could be performed in the text. This was done with some but not all of the written problems. In the preface Adams states "To answer the several intentions of this work, it will be necessary that it should be put into the hands of every Arithmetician: the blank after each example is designed for operation by the scholar, which being first wrought upon slate or waste paper, he may afterwards transcribe into his book." It should be recalled that this book came at a time when most of the arithmetic or *cyphering* in schools was done on slates and in copybooks. Texts were not commonly owned by all the pupils. Some moderns may object to recopying the same exercise after once having done it on slate or waste paper. However, transcribing work can serve as added practice to help establish a procedure, it provides opportunity for improving the form in setting down the work, and it serves as an opportunity for checking.

In carrying out the workbook spacing in printing, Adams did a fine job in featuring the advantages of Federal Money over British Money. He did this by printing a double column page with a problem in British Money on the left and one with Federal Money on the right. Then he illustrated the necessary work and showed that Federal Money was not only easier to use in figuring but required only half the space.

In his "Directions to the Scholar," he advises, "Remember that youth, like the morning, will soon be past, and that opportunities once neglected, can never be regained." From our experiences of the twentieth century, we might rather advise that opportunity once neglected probably will not be regained. Last week while watching college freshmen take an examination, the following errors were observed, $49+5=64$, 60 billion was written 60,000,000; $2.9\%=2.9$; $39\times 6=224$; $0.96\times 2000=192{,}000$; $54-18=38$; etc. It is not so many years ago that we were hearing the "Philosopher of Education" saying, "If Johnny ever needs to use multiplication when he grows up, he will be able to learn it in a short time when he needs it." Not many teachers who teach above the sixth grade in fields other than Educational Theory would now ascribe to that doctrine. Author Adams advises further, "As much as possible, endeavor to do everything of yourself; *one* thing found out by your own thought and reflection, will be of more real use, than *twenty* things told you by an instructor." And again, "Understand everything as you go along." The writer's observations during the past fifteen years tend to the conclusion that pupils are growing less self reliant and want to be told and shown every step. These procedures of course defeat the pur-

pose of education, i.e., to make the student a rational, self reliant, and functioning human being when he encounters a mathematical situation. What teacher hasn't been distressed by the complaint, "But that isn't exactly the way he had it in class." Adams wanted his scholars to develop independence of attack based upon understanding of the process and its usefulness.

After each of the several sections of the book a "supplement" is inserted. This consists of questions upon the nature and use of the previous materials and more complicated problems. Perhaps this was Adams' way of caring for the brighter pupils as well as a means of developing meanings and significance. He asks such questions as (1) "Why do you carry for ten rather than any other number?" (2) "How does it appear that in subtracting a less number from a greater, the occasional *borrowing of ten* does not affect the difference between these two numbers?" (3) "What is understood by more requiring less and less requiring more?" (Refers to rule of three.)

It is interesting to note that in both the Tenth and Sixteenth Yearbooks of the National Council of Teachers of Mathematics, we find such statements as "Children should understand and see the sense of what they are doing." Adams wanted this 125 years ago. He advised, "These questions the pupil should be made to study and reflect upon, till he can of himself devise the proper answer."

The closing sections of Adams' book are particularly suggestive because of their content and treatment. Section III has approximately 50 pages devoted to "Rules Occasionally Useful to Men in Particular Callings and Pursuits of Life." Obviously, this was not intended to be studied in the same manner as the regular textual material. In terms of the time at which this book was written, this material could have been used as supplementary or extensional materials for the brighter scholars, it was useful for scholars having special interests and served as a reference book for laymen. It is interesting to note that square and cube root are found in this section with a treatment " . . . of the reason and nature of the operations . . . ." It is only in the twentieth century that square and cube root again became optional materials. Other topics treated in this section III are Fellowship, Barter, Loss and Gain, Duodecimals, Measuring Wood, Painter's and Joiner's Work, Gauging, Mechanical Powers, Alligation, Position, etc. Practically all of these topics were for many years part of the standard course in arithmetic. To many teachers who are unfamiliar with the development of arithmetic as a subject for study, such terms as Alligation and Double Position are more puzzling than intriguing.

Daniel Adams included the usual lists of "Miscellaneous Questions" and "Diverting Questions." In his day authors were not burdened with the modern insistence upon reality and social utility in problems, e.g. " . . . what would be the weight of such a tube, which would extend across the Altantic from Boston to London, estimating the distance to be 3000 miles?" "The Frog and the Well," "The Hundred Geese" and similar puzzle problems that date back to antiquity are included.

Of special note is the treatment of Fractions. In the first part of the text there is a simple one-page explanation of "Vulgar Fractions" with this statement, "The arithmetic of Vulgar Fractions is tedious and even intricate to beginners. Besides they are not of necessary use. We shall not therefore enter into any further consideration of them here." So Adams proceeds directly to the presentation of Decimal Fractions followed by Federal Money. Then in a final appendix in the book, he gives an eight-page treatment of Vulgar Fractions. This view of fractions is not unique with Adams. Several authors including Erastus Root had previously dropped Vulgar Fractions. Chauncey Lee

in his American Accountant (1797) relates "The use of vulgar (common) fractions may be advantageously superseded by that of decimals, they are viewed as an unnecessary branch of common school education and therefore omitted from this compendium." Adams recognized the impossibility of deleting all common fractions. In the usual supplement following the topic of fractions (decimal fractions) he shows how to handle computations with such common fractions as halves and fourths by converting them to equivalent decimal fractions. Again it is interesting to note that several writers of the twentieth century have urged this procedure. The chief impetus for dropping fractions was the adoption of Federal Money in decimal proportion by Congress in 1786. It was a confusing monetary situation in the early nineteenth century. State currencies, Federal Money, and British Exchange were all used simultaneously. Not until about 1840 was the situation entirely resolved in favor of Federal Money.

Daniel Adams used a number of salutary ideas more than 125 years ago. Books by other authors such as Daboll, Pike, Colburn, Root, etc., also are worth restudy. You will find them, if you are fortunate, in book stalls, second-hand stores, antique shops, and at country auctions.

Jefferson as a Scientist and Inventor

Library of Congress to Portray Jefferson as a Man of Science in Special Exhibition Marking the Bicentennial of His Birth

NONE of his contemporaries, with the possible exception of Benjamin Franklin, had as varied an interest in the pursuit of science as had Thomas Jefferson. Yet, despite much evidence of his advanced scientific knowledge and of his application of science to the common pursuits of life, the story of Jefferson's accomplishments as a man of science and inventor never has become widely known.

It is especially appropriate therefore that one of the exhibitions opening at the Library of Congress on Monday, April 12, preliminary to the Jefferson Bicentennial Celebration the following day, will be a display designed to reveal and evaluate Jefferson as a scientist. The cases housing this exhibition will be located just outside the Thomas Jefferson Room for Science and Technology, on the fifth floor of the Annex Building, where Ezra Winter's murals in memory of Jefferson are located.

Of foremost importance among Jefferson's scientific achievements was his work as a statesman which laid the foundations for several of the Federal scientific bureaus of today. An item of unusual interest in the display is his report as Secretary of State on the subject of establishing uniformity in weights, measures and coins in the United States. This document, dated July 4, 1790, became the basis of that interest on the part of the Government which resulted in the establishment of the Bureau of Standards. Other material to be exhibited will indicate Jefferson's part in laying the foundations of the American patent system and patent laws, in instituting in 1807 the Coast Survey (continued today as the U. S. Coast and Geodetic Survey), and in encouraging the early steps leading to the eventual establishment of the Naval Observatory and Hydrographic Office.

Photographs of a number of Jefferson's inventions, showing his practical application of the principles of science, also will be on display. Jefferson's work as the Father of American Paleontology also will be illustrated by photographs of some of the fossil bones collected by Jefferson which are now on view in American Museums. An original of his monograph on the subject of fossils also will be shown.

Another interesting section of the exhibition will consist of representative selections from Jefferson's own outstanding collection of books on science. Many of these rare volumes bear indications of extensive use by Jefferson and some are themselves important sources of information about Jefferson's scientific knowledge since they contain supplementary notes, corrections, and other manuscript insertions by Jefferson. Even more important clews to Jefferson's scientific mind will be shown through selections from his many letters to scholars and scientists of his era. Letters from Jefferson dealing with most of the different fields of science will be included, as well as his letter to the officers of the American Philosophical Society acknowledging his election as president.

Another letter of extreme importance and interest was sent by Jefferson to Benjamin Banneker, a Negro mathematician and astronomer of the last part of the 18th century. A manuscript draft of this letter, in which Jefferson acknowledges a copy of Banneker's almanac and expresses his interest in and sympathy for the Negro race, also will be shown. Jefferson compliments Banneker for his talent and expresses the hope that the condition of the Negro will be improved.

Representative articles giving present-day evaluations of Jefferson as a scientist will complete the exhibit which will be of great interest.

"Handbook on Education and the War" Issued

PUBLICATION of a comprehensive "Handbook on Education and the War" was announced recently by the U. S. Office of Education. Based on the proceedings of the National Institute on Education and the War, the "Handbook" is an over-all survey of the major wartime problems of education.

The 359-page "Handbook" is divided into two parts, one containing the full text of statements by heads of those Federal war agencies which touch education, and the other part containing reports of symposiums held on 26 of the most acute wartime educational issues. The 26 key problems are grouped under 4 general headings: *Training Manpower*, *School Volunteer War Service*, *Curriculum in Wartime*, and *Financing Education in Wartime*.

The Picnic Problem

By A. R. JERBERT
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TEACHERS will readily recall the fundamental 4-step procedure which elementary algebra supplies for solving any and every kind of problem which presents itself.

1. Single out one of the unknown quantities and denote it by x.
2. Express the remaining quantities in terms of x.
3. From the conditions posed by the problem form an equation.
4. Solve this equation.

With repeated practice associational habit invests this routine with a sequential consecutive, character such that each step automatically calls forth the next one.

If, therefore, students can be conditioned to confront each new problem with a readiness to "let x equal," success in problem solving is assured.

Thorough conditioning requires, however, a surprising amount of time and effort. Even mathematics teachers who have the advantage which comes from repeatedly explaining and emphasizing this procedure find that they are slow to adopt it when confronted with a new type problem.

The writer was therefore very much interested, recently, when a student in his teachers' course volunteered the remark that "with her this procedure had become a confirmed habit."

As evidence she cited a problem which she had encountered in connection with a picnic lunch. The question was to divide 11 canteloupes among 27 people. Division into halves and thirds was indicated. But how many of each?

She was particularly pleased by the fact that she found herself spontaneously letting x equal the number to be halved and $11-x$ the number to be divided into thirds. This led to the equation

$$2x+3(11-x)=27,$$

whence

$$x=6,\ 11-x=5.$$

To generalize the problem it is clear that we might equally well have a canteloupes for b people, $b>a$. The division correspondingly could be made into cth's and dth's with $d=c+1$.

This leads to the equation:

$$x(c+1)+(a-x)c=b,$$

whence

$$c=\frac{b-x}{a}\cdot$$

If we divide b by a we shall obtain a quotient q and a remainder r $(r<a)$.

Hence $b=qa+r$.

Substituting we obtain,

$$c=\frac{b-x}{a}=q+\frac{r-x}{a}\cdot$$

and since (c) is integral we must have $x=r+ka$ (k any integer). But since we also have $x<a$, it follows that $k=0$ and $x=r$, whence $c=q$.

The equation $b=qa+r$ becomes therefore $b=ca+x$.

In other words if we divide b by a the quotient c is the smaller of the two division numbers and the remainder x is the number (of canteloupes) which will have to be divided into $(c+1)$th's.

For example with 87 people and 21 canteloupes we have,

$$87=4(21)+3.$$

This means that 3 canteloupes are to be divided into fifths and the remaining 18 into fourths.

In similar fashion any division identity may be interpreted as the solution of a picnic problem which requires the (approximately) equitable division of cakes and canteloupes.

EDITORIALS

Concerning Subscriptions That Expire in May

Any one who began his subscription to The Mathematics Teacher with the October issue in 1942 is automatically a member of The National Council of Teachers of Mathematics until October, 1943. However, since no issues of the magazine are published in June, July, August, and September of 1943, those who paid the membership fee of $2.00 in October of last year should send in their renewals before October, 1943 in order to save the Council inconvenience and loss of money. Costs of publication are rising and in order not to have to raise the price of the journal during the emergency we bespeak the cooperation of our members in being prompt in making renewals. In order to make sure that this matter is not overlooked, The Mathematics Teacher will send out early in April, if not before, cards to all members whose subscriptions expire in May (even though their membership runs to October) and it is hoped that members will be prompt in renewing membership so as to facilitate matters in the office. It has been almost impossible to plan for the October issue each year because members are so careless about renewing in time. Moreover, entirely too many members fail to renew at all even though two or three notices and a personal appeal from the Editor have been sent out. It is more important now than ever before that every present member "stick by the ship" if we are to weather the storms ahead and if the Council is to continue to do its work effectively. We can no longer send an extra copy of the magazine to members who fail to renew on time. Increased costs of printing make our previous practice impossible.

W. D. R.

Who Should Study Mathematics?

In these days when one hears so much about the importance of mathematics in the education of American citizens, teachers of mathematics should begin to study and discuss the question as to who should study mathematics and how far its study should be pursued by high school pupils.

There is no question but that high school pupils of ability and interest in the subject should be encouraged to study mathematics for four years. Even those of so called *average* ability or above who like mathematics should be encouraged to do so. Moreover, those pupils who know that they are going into lines of work where a knowledge of mathematics is basic, as in engineering, should be encouraged to continue their study of mathematics as long as they are in school. Our recent awakening to the fact that many pupils have been deprived of the study of mathematics which they actually needed is evidence of our lethargy in the matter.

However, there is a tendency now in some places to try to force many pupils to study mathematics beyond their need and ability. It is to be hoped that guidance in such matters will be carefully discussed and planned for all schools. It would seem better that we have two years of carefully planned general mathematics in the high school for most pupils and that the door be left open from there on only for those whose need, ability, or interest impels them to continue.

All of this means, of course, that better organization and methods of teaching mathematics are now more important than ever, if mathematics is to continue to occupy the position of high importance which it now holds.

W. D. R.

◆ IN OTHER PERIODICALS ◆

By NATHAN LAZAR

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The American Mathematical Monthly

January 1943, vol. 50, no. 1.

1. Menger, Karl, "What Is Dimension?" pp. 2–7.
2. Price, G. B., "Adjustments in Mathematics to the Impact of War," pp. 31–34.
3. Niven, Ivan, "On Elliptic Integrals," pp. 41–42.
4. Bruce, J. M., "Approximation to a Central Angle," pp. 43–45.
5. Frame, J. S., "Some Applications of the Slide Rule," pp. 55–57.

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December 1942, vol. 17, no. 3.

1. Kenny, J. F., "Characteristic Functions in Statistics," pp. 99–114.
2. Dorwart, H. L., "Values of the Trigonometric Ratios of $\pi/5$ and $\pi/10$," pp. 115–116.
3. Fettis, H. E., "On Various Methods of Solving Cubic Equations," pp. 117–130.

School Science and Mathematics

February 1943, vol. 42, no. 2.

1. Hart, William L., "The Nation Calls for Mathematics," pp. 105–116.
2. Havighurst, Robert J., and others, "High School Science and Mathematics in Relation to the Manpower Problem," pp. 127–157.
3. Nyberg, Joseph A., "Notes from a Mathematics Classroom," pp. 158–161.

Miscellaneous

1. Carpenter, I. M., "Present Status of Mathematics in the High School," *Texas Outlook*, 27: 37–38, January 1943.
2. Doyle, H. G., "Educationist Inferiority Complex," *Hispania*, 25: 450–451, December 1942.
3. Hall, J. V., "Oral Aids to Problem Solving, *Elementary School Journal*, 43: 220–224, December 1942.
4. Lindsay, R. B., "Physics and Mathematics in the War Training Program at Brown University," *American Journal of Physics*, 10: 316–319, December 1942.
5. Morse, M., "Mathematics and the Maximum Scientific Effort in Total War," *Scientific Monthly*, 56: 50–55, January 1943.
6. Moskowitz, D. H., "Mathematics Teachers and the Pre-Induction Program," *High Points*, 24: 5–9, December 1942.
7 Paulin, E. A., "What Implications for the High School Program in Mathematics Are To Be Drawn from the War Program?" *Proceedings, National Catholic Education Association*, 1942: 381–383.
8. Risden, G. A., "Gaining Number Concepts," *Instructor*, 52: 22, February 1943.

THE NATIONAL INSTITUTE on Education and the War, held last autumn under the sponsorship of the U. S. Office of Education Wartime Commission, was attended by over 700 of the Nation's education leaders, from every State in the Union.

In the Foreword to the "Handbook," John W. Studebaker, U. S. Commissioner of Education, says: "Because it represents the best wartime thinking of so many alert minds, it should prove a useful guide to every educator in intensifying efforts to win the war."

Copies are available from the Superintendent of Documents, Washington, D. C., at 55 cents each.

The 26 areas of educational war service presented are: TRAINING MANPOWER—1. What Can Be Done to Solve Problems of Maintaining Teaching Staffs? 2. In What Directions Shall Pre-employment and In-Service Training of Technicians Be Further Extended? 3. How Shall Colleges and Universities Assist the Army and the Navy in Their Specialist Training Programs? 4. What Can the Schools Do for the Adult Illiterate and Alien in Wartime? 5. What Program of Training and Adjustment for Adults Will Be Needed in the Post-War Years? 6. How May Secondary Schools and Junior Colleges Train Workers Needed by the Military Forces and War Industries? 7. How Shall Workers in the Food for Freedom Program Be Secured and Trained? 8. How and Where Shall We Obtain the Trained People and Facilities Needed for Extending School Services to Children Whose Mothers or Parents Are Employed in Activities Related to the War Effort?

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